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ON GLOBAL EXISTENCE AND TREND TO THE EQUILIBRIUM FOR THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM WITH EXTERIOR CONFINING POTENTIAL

by

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Abstract. — We prove a global existence result with initial data of low regularity, and prove the trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with small non linear term but with a possibly large exterior confining potential in dimension $d = 2$ and $d = 3$. The proof relies on a fixed point argument using sharp estimates (at short and long time scales) of the semi-group associated to the Fokker-Planck operator, which were obtained by the first author.

1. Introduction and results

1.1. Presentation of the equation. — Let $d = 2$ or $d = 3$. We consider the Vlasov-Poisson-Fokker-Planck system (VPFP for short) with external potential, which reads, for $(t, x, v) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$(1.1) \qquad \begin{cases} \partial_t f + v.\partial_x f - (\varepsilon_0 E + \partial_x V_e).\partial_v f - \gamma\partial_v.\left(\partial_v + v\right) f = 0, \\ E(t,x) = -\dfrac{1}{|\mathbb{S}^{d-1}|}\dfrac{x}{|x|^d} \star_x \rho(t,x), \qquad \text{where} \quad \rho(t,x) = \displaystyle\int f(t,x,v)dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $x \mapsto V_e(x)$ is a given smooth confining potential (see Assumption 1 below). The constant $\varepsilon_0 \in \mathbb{R}$ is the total charge of the system and in the sequel we assume that either $\varepsilon_0 > 0$ (repulsive case) or $\varepsilon_0 < 0$ (attractive case) in the case $d = 3$. The constant $\gamma > 0$ is the friction-diffusion coefficient, and for simplicity we will take $\gamma = 1$.

The unknown f is the distribution function of the particles. We assume that $f_0 \geq 0$ and that $\displaystyle\int f_0(x,v)dxdv = 1$, it is then easy to check that once a good existence theory is given, these properties are preserved, namely that for all $t \geq 0$

$$f \geq 0 \qquad \text{and} \qquad \int f(t,x,v)dxdv = 1,$$

and we refer to Section 3.1 for more details and other basic results.

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This equation is a model for a plasma submitted to an external confining electric field (in the repulsive case) and also a model for gravitational systems (in the attractive case). When there is no external potential ($V_e = 0$), the equation has been exhaustively studied. First existence results were obtained by Victory and O'Dwyer in 2d [**23**] and by Rein and Weckler [**26**] in 3d for small data. Bouchut [**2**] showed that the equation is globally well-posed in 3 dimensions using the explicit kernel. The long time behavior (without any rate) has been studied with or without external potential by Bouchut and Dolbeault in [**3**], Carillo, Soler and Vazquez [**4**], and also by Dolbeault in [**10**].

When there is a confining potential, arbitrary polynomial trend to the equilibrium was established in [**7**] where a first notion of hypocoercivity [**29**] was developed and used later to the full model [**8**]. The exponential trend to the equilibrium was shown in the linear case (the Fokker-Planck equation) for a general external confining potential in [**18**] (see also [**16**]). So far, in the non-linear case, there is no general result about exponential trend to the equilibrium. In the case of the torus (and $V = 0$), the strategy of Guo can be applied to many models (see e.g. [**12, 13, 14**]). In the case when the potential is explicitly given by $V_e(x) = C|x|^2$, a recent result with small data is given in [**20**], following the micro-macro strategy of Guo.

In all previous cases (torus, $V_e = 0$ or polynomial of order 2), mention that one can compute explicitly the Green function of the Fokker-Planck operator and also that exact computations can be done thanks to vanishing commutators. Here instead we will rely on estimates (in short and long time) of the linear solution of the Fokker-Planck operator obtained by the first author in [**17**], and our approach allows us to deal with a large class of confining potentials V_e. Indeed, in [**17**, Theorem 1.3] a first exponential trend to the equilibrium result for a VPFP type model was given, but only for a mollified non-linearity. We will prove here a global existence result in the full VPFP case, with trend to equilibrium assuming that the initial condition f_0 is localised and has some Sobolev regularity and under the assumption that the electric field is perturbative in the sense that $|\varepsilon_0| \ll 1$.

Let us now precise our notations and hypotheses. We do not try to optimise the assumptions on the confining potential V_e and first assume the following

Assumption 1. — *The potential $x \mapsto V_e(x)$ satisfies*

$$e^{-V_e} \in \mathcal{S}(\mathbb{R}^d), \quad \textit{with} \quad V_e \geq 0 \quad \textit{and} \quad V_e'' \in W^{\infty,\infty}(\mathbb{R}^d).$$

Observe that the assumption $V_e \geq 0$ can be relaxed by assuming that V_e is bounded from below and adding to it a sufficiently large constant.

We now introduce the Maxwellian of the equation (1.1)

$$\mathcal{M}_\infty(x, v) = \frac{e^{-(v^2/2+V_e(x)+\varepsilon_0 U_\infty(x))}}{\int e^{-(v^2/2+V_e(x)+\varepsilon_0 U_\infty(x))} dx dv}, \tag{1.2}$$

where U_∞ is a solution of the following Poisson-Emden type equation

$$-\Delta U_\infty = \frac{e^{-(V_e+\varepsilon_0 U_\infty)}}{\int e^{-(V_e(x)+\varepsilon_0 U_\infty(x))} dx}. \tag{1.3}$$

Actually, one gets that under Assumption 1 and $|\varepsilon_0|$ small enough (assuming additionally that $\varepsilon_0 > 0$ in the case $d = 2$), the equation (1.3) has a unique (Green) solution U_∞ which belongs to $W^{\infty,\infty}(\mathbb{R}^d)$ uniformly w.r.t $|\varepsilon_0|$ (see Propositions 3.5 and 3.6 following results from [**9**]). The Maxwellian $\mathcal{M}_\infty$ is then in $\mathcal{S}(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and is the unique L^1-normalised steady solution of equation (1.1).

In the case $d = 2$ and $\varepsilon_0 < 0$, existence and uniqueness of solutions to (1.3) are unclear, that's why we do not consider this case.

For convenience, we now introduce the effective potential at infinity

$$V_\infty \overset{\text{def}}{=} V_e + \varepsilon_0 U_\infty \quad \text{so that} \quad \mathcal{M}_\infty(x,v) = \frac{e^{-(v^2/2+V_\infty(x))}}{\int e^{-(v^2/2+V_\infty(x))}dxdv}. \tag{1.4}$$

The second assumption on V_e is the following

Assumption 2. — *The so-called Witten operator $W = -\Delta_x + |\partial_x V_e|^2/4 - \Delta_x V_e/2$ has a spectral gap in $L^2(\mathbb{R}^d)$. We denote by $\kappa_0 > 0$ the minimum of this spectral gap and $d/2$.*

Example 1.1. — As an example, we can check that if V_e satisfies Assumption 1 and is such that

$$|\partial_x V_e(x)| \underset{|x|\longrightarrow\infty}{\longrightarrow} +\infty$$

then it satisfies also Assumption 2 since it has a compact resolvent.

We introduce now the functional framework on which our analysis is done. We consider the weighted space B built from the standard L^2 space after conjugation with a half power of the Maxwellian

$$B \overset{\text{def}}{=} \mathcal{M}_\infty^{1/2} L^2 = \big\{ f \in \mathcal{S}'(\mathbb{R}^{2d}) \text{ s.t. } f/\mathcal{M}_\infty \in L^2(\mathcal{M}_\infty dxdv) \big\}. \tag{1.5}$$

We define the natural scalar product

$$\langle f, g\rangle = \int f g \mathcal{M}_\infty^{-1} dxdv,$$

and the corresponding norm

$$\|f\|_B^2 = \langle f, f\rangle = \int f^2 \mathcal{M}_\infty^{-1} dxdv.$$

Next, consider the Fokker-Planck operator associated to the potential V_∞ defined by

$$K_\infty = v.\partial_x - \partial_x V_\infty(x).\partial_v - \gamma \partial_v.\left(\partial_v + v\right). \tag{1.6}$$

The last object we need before writing our equation in a suitable way is the limit electric field

$$E_\infty(x) = \partial_x U_\infty(x) = -\frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star_x \int \mathcal{M}_\infty(x,v)dv.$$

With all the previous notations, the VPFP equation (1.1) can be rewritten

$$\begin{cases} \partial_t f + K_\infty f = \varepsilon_0 (E - E_\infty)\partial_v f, \\ E(t,x) = -\dfrac{1}{|\mathbb{S}^{d-1}|}\dfrac{x}{|x|^d} \star_x \rho(t,x), \qquad \text{where} \quad \rho(t,x) = \displaystyle\int f(t,x,v)dv, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1.7}$$

We define the operator

$$\Lambda_x^2 = -\partial_x.\big(\partial_x + \partial_x V_\infty\big) + 1$$

which is up to a conjugation with $\mathcal{M}_\infty^{1/2}$ the Witten operator introduced in Assumption 2 but defined on B, and

$$\Lambda_v^2 = -\partial_v.(\partial_v + v) + 1,$$

which is again up to a conjugation the harmonic oscillator in velocity. They both are non-negative selfadjoint unbounded operators in B. We also introduce

$$\Lambda^2 = -\partial_x.\big(\partial_x + \partial_x V_\infty\big) - \partial_v.(\partial_v + v) + 1 = \Lambda_x^2 + \Lambda_v^2 - 1.$$

It is clear that
$$1 \leq \Lambda_x^2,\ \Lambda_v^2 \leq \Lambda^2.$$
As we mentioned previously, if V_e satisfies Assumptions 1 and 2, then $V_\infty = V_e + \varepsilon_0 U_\infty$ also does, and we check in Subsection 3.3 that the operator
$$-\partial_x.\big(\partial_x + \partial_x V_\infty\big) - \partial_v.(\partial_v + v) = \Lambda^2 - 1$$
has 0 as single eigenvalue and a spectral gap bounded in B which is, uniformly w.r.t $|\varepsilon_0|$ small, bounded from below by $\kappa_0/2$.

In the sequel, we will need the anisotropic chain of Sobolev spaces: for $\alpha, \beta \geq 0$
$$B^{\alpha,\beta} = B^{\alpha,\beta}_{x,v}(\mathbb{R}^{2d}) = \big\{ f \in B\ :\ \Lambda_x^\alpha f \in B \ \text{ and } \ \Lambda_v^\beta f \in B \big\}, \tag{1.8}$$
and we endow this space by the norm
$$\|f\|_{B^{\alpha,\beta}} = \|\Lambda_x^\alpha f\|_B + \|\Lambda_v^\beta f\|_B.$$
In the case $\alpha = \beta$ we simply define
$$B^\alpha = B^{\alpha,\alpha}_{x,v}(\mathbb{R}^{2d}) = \big\{ f \in B\ :\ \Lambda^\alpha f \in B \big\},$$
with the norm
$$\|f\|_{B^\alpha} = \|\Lambda^\alpha f\|_B \sim \|f\|_{B^{\alpha,\alpha}}.$$
We observe that $\mathcal{M}_\infty \in B^{\alpha,\beta}$ for all $\alpha, \beta \geq 0$, since we have $\mathcal{M}_\infty \in \mathcal{S}(\mathbb{R}^{2d})$.

1.2. Main results. — We are now able to state our global well-posedness results.

Theorem 1.2. — *Let $d = 2$ and let $f_0 \in B(\mathbb{R}^4)$. Assume moreover that Assumptions 1 and 2 are satisfied. Then if $\varepsilon_0 > 0$ is small enough, there exists a unique global mild solution f to* (1.1) *in the class*
$$f \in \mathcal{C}\big([0, +\infty[\,;\, B(\mathbb{R}^4) \big).$$
Moreover, the following convergence to equilibrium holds true
$$\|f(t) - \mathcal{M}_\infty\|_B \leq C_0 e^{-\kappa_0 t/c}, \qquad \forall t \geq 1,$$
and
$$\|E(t) - E_\infty\|_{L^\infty(\mathbb{R}^2)} \leq C_1 e^{-\kappa_0 t/c}, \qquad \forall t \geq 1.$$

By mild, we mean f and E which satisfy the integral formulation of (1.7), namely
$$\begin{cases} f(t) = e^{-K_\infty} f_0 + \varepsilon_0 \displaystyle\int_0^t e^{-(t-s)K_\infty} (E(s) - E_\infty) \partial_v f(s) ds, \\ E(t) = -\dfrac{1}{|\mathbb{S}^{d-1}|} \dfrac{x}{|x|^d} \star_x \displaystyle\int f(t) dv. \end{cases} \tag{1.9}$$

In the case $d = 3$, we need to assume more regularity on the initial condition, but the known results about the uniqueness of the Poisson-Emden equation (see Subsection 3.2) allow to consider also the case $\varepsilon_0 < 0$.

Denote by
$$U_0 = \frac{1}{4\pi|x|} \star_x \int f_0 dv, \tag{1.10}$$
which is such that $\Delta U_0 = \displaystyle\int f_0 dv$. Then

Theorem 1.3. — *Let $d = 3$ and $1/2 < a < 2/3$. Assume that $f_0 \in B^{a,a}(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ is such that $U_0 \in W^{2,\infty}(\mathbb{R}^3)$. Assume moreover that Assumptions 1 and 2 are satisfied. Then if $|\varepsilon_0|$ is small enough, there exists a unique global mild solution f to* (1.1) *in the class*

$$f \in \mathcal{C}\big([0,+\infty[\,;\, B^{a,a}(\mathbb{R}^6)\big) \cap L^\infty_{loc}\big([0,+\infty[\,;\, L^\infty(\mathbb{R}^6)\big).$$

Moreover, for all $a \le \alpha < 2/3$ and $a \le \beta < 1$ such that $3\alpha - 1 < \beta < 1$

$$f \in \mathcal{C}\big(]0,+\infty[\,;\, B^{\alpha,\beta}(\mathbb{R}^6)\big), \tag{1.11}$$

and the following convergence to equilibrium holds true

$$\|f(t) - \mathcal{M}_\infty\|_{B^{\alpha,\beta}} \le C_0 e^{-\kappa_0 t/c}, \qquad \forall t \ge 1,$$

and

$$\|E(t) - E_\infty\|_{L^\infty(\mathbb{R}^3)} \le C_1 e^{-\kappa_0 t/c}, \qquad \forall t \ge 1.$$

In the previous lines, the constants $c, C_1, C_2 > 0$ only depend on $\|V_\infty\|_{W^{2,\infty}}$ where V_∞ was defined in (1.4), on $\|U_0\|_{W^{2,\infty}}$ and on f_0.

Notice that in Theorem 1.3, the parameters (α, β) can be chosen independently from a. It is likely that the assumption $a < 2/3$ is technical, but our proof needs that $\beta < 1$ (see e.g. Corollary 2.6). Since in this work we focus on low regularity issues, we did not try to relax this hypothesis.

It is likely that the assumption made on U_0 is technical. It is needed here in order to guarantee that the linearised equation near $t = 0$ enjoys reasonable spectral estimates. Observe (see Remark 3.17 for more details), that the assumption $f_0 \in B^{a,a}(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ alone ensures that $U_0 \in W^{2,p}(\mathbb{R}^3)$ for any $2 \le p < +\infty$.

An analogue of the regularizing estimate (1.11) can also be obtained in Theorem 1.2. This can be proven by getting estimates in some spaces $B^{\alpha,\beta}_{x,v}$ as in the proof of Theorem 1.3 (see Section 5). We did not include it here in order to simplify the argument.

The proof uses estimates of e^{-tK_∞} in the space B, obtained in [**17**] by the first author. Theorem 1.3 extends [**17**, Theorem 1.3] where he considered a regularised version of the electric field E in (1.1), which was so that $E(t) \in L^\infty(\mathbb{R}^3)$ for any $f \in B$. Here we tackle this difficulty by using the Sobolev regularity of f and a gain given by the integration in time. The proof relies on a fixed point argument in a space based on $B^{\alpha,\beta}$ in the (x, v) variables, and allowing an exponential decay in time.

As a consequence of Theorems 1.2 and 1.3, we directly obtain the exponential decay of the relative entropy. Let us define

$$H(f(t), \mathcal{M}_\infty) = \iint f(t) \ln\Big(\frac{f(t)}{\mathcal{M}_\infty}\Big) dx dv,$$

then

Corollary 1.4. — *Let $d = 2$ or $d = 3$. Then under the assumptions of Theorem 1.2 or Theorem 1.3, the corresponding solution f of* (1.1) *satisfies*

$$0 \le H(f(t), \mathcal{M}_\infty) \le C e^{-\kappa t/c},$$

where $C, c > 0$ only depend on second order derivatives of $V_e + \varepsilon_0 U_\infty$ and on f_0.

We refer to [**17**, Corollary 1.4] for the proof of this result.

1.3. Notations and plan of the paper. —

Notations. — *In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.*

The rest of the paper is organised as follows. In Section 2 we prove some linear estimates on e^{-tK} (where K is a generic linear Fokker-Planck operator). In Section 3 we gather some estimates on solutions of (1.1). Finally, Sections 4 and 5 are devoted to the proofs of Theorems 1.2 and 1.3 with fixed points arguments.

2. Semi-group estimates

In this section, we denote by V a generic potential satisfying Assumptions 1 and 2. We also denote by K the associated generic linear Fokker-Planck operator

$$K = v.\partial_x f - \partial_x V.\partial_v - \partial_v.(\partial_v + v).$$

Similarly, the operators $\Lambda_x^2 = -\partial_x(\partial_x + \partial_x V) + 1$, $\Lambda^2 = \Lambda_x^2 + \Lambda_v^2 - 1$, the normalized Maxwellian $\mathcal{M}(x, v) = e^{-(V(x)+v^2/2)}$ and spaces of type $B^{\alpha,\beta}$ are built with respect to this generic potential V. For convenience, we also denote by $X_0 = v.\partial_x f - \partial_x V.\partial_v$.

The aim of this section is to state some estimates of e^{-tK} in $B-$type norms. These are consequences of [**17**]. In all the following we pose

$$\kappa = \kappa_0/C_0,$$

where κ_0 is the spectral gap of the operator W defined in Assumption 2 (with V as a potential), and C_0 is a large constant depending only on derivatives of V'' explicitly given in [**18**, Theorem 0.1].

The operator K is maximal accretive in B (see e.g. [**16**, Theorem 5.5]). This enables us to define e^{-tK} and to prove that

$$\big\|e^{-tK}\big\|_{B\to B} \leq 1. \tag{2.1}$$

Following [**18**, Theorem 3.1], operator $\mathrm{e}^{-tK} \longrightarrow Id$ when $t \longrightarrow 0$, strongly in $B^{a,a}$ for any $a \geq 0$. Observe that all the estimates in this section are independent of the dimension d. For a complete analysis of the linear Fokker-Planck operator we refer to [**18**] or [**16**]. We now give some regularizing estimates for the semi-group associated to K, in the spirit of [**17**, Section 3].

Proposition 2.1. — *There exists $C > 0$ so that for all $\alpha, \beta \in [0, 1]$ and all $t > 0$*

$$\|\Lambda_x^\alpha e^{-tK}\|_{B\to B} \leq C(1 + t^{-3\alpha/2}), \quad \|e^{-tK}\Lambda_x^\alpha\|_{B\to B} \leq C(1 + t^{-3\alpha/2}) \tag{2.2}$$

and

$$\|\Lambda_v^\beta e^{-tK}\|_{B\to B} \leq C(1 + t^{-\beta/2}), \quad \|e^{-tK}\Lambda_v^\beta\|_{B\to B} \leq C(1 + t^{-\beta/2}). \tag{2.3}$$

In the previous bounds, the constant C only depends on a finite number of derivatives of V.

Remark 2.2. — Note that the exponents $1/2$ in (2.3) and $3/2$ in (2.2) when $\alpha = 1$ are optimal at least in the case $V = 0$ and in the case when V is a definite quadratic form in x. This can be checked since in these both cases, the Green kernel of e^{-tK} is explicit. In the case $V = 0$ we refer to [**2**], and when V is quadratic, we refer to the general Mehler formula given in [**19**, Section 4].

Proof of Proposition 2.1. — We first prove the estimate (2.3). In [**17**, Proposition 3.1], reinterpreted in our framework, reads

$$\left\|(\partial_v + v)e^{-tK}\right\|_{B\to B} \leq C(1+t^{-1/2}). \tag{2.4}$$

For f a solution of the equation

$$\partial_t f + Kf = 0, \quad f(t=0) = f_0,$$

with normalized initial condition $f_0 \in \mathcal{C}_0^\infty$, and using the regularization property of e^{-tK}, we have for $t > 0$

$$\begin{aligned} \|\Lambda_v f(t)\|^2_{B\to B} &= \langle \Lambda_v^2 f(t), f(t)\rangle \\ &= \|(\partial_v + v) f(t)\|^2_{B\to B} + \|f(t)\|^2_{B\to B} \\ &\leq C(1+t^{-1/2})^2 + 1 \\ &\leq C'(1+t^{-1/2})^2. \end{aligned}$$

Using that $B^{0,0}_{x,v} = B$ and (2.1), we therefore have that

$$\left\|e^{-tK}\right\|^2_{B\to B^{0,1}_{x,v}} \leq C'(1+t^{-1/2}), \quad \left\|e^{-tK}\right\|^2_{B\to B^{0,0}_{x,v}} \leq C''$$

and by interpolation we get that for all $0 \leq \beta \leq 1$

$$\left\|e^{-tK}\right\|^2_{B\to B^{0,\beta}_{x,v}} \leq C(1+t^{-1/2})^\beta \leq C_b(1+t^{-\beta/2})$$

which reads

$$\|\Lambda_v^\beta \mathrm{e}^{-tK}\|_{B\to B} \leq C_\beta(1+t^{-\beta/2})$$

which is the first result. For the converse estimate, we use that K^*, the adjoint of K in B given by $K^* = -X_0 - \partial_v.(\partial_v + v)$, has the same properties as K so that for all $t > 0$,

$$\|\Lambda_v^\beta \mathrm{e}^{-tK^*}\|_{B\to B} \leq C'_\beta(1+t^{-\beta/2}).$$

Taking the adjoints of this yields

$$\|\mathrm{e}^{-tK}\Lambda_v^\beta\|_{B\to B} \leq C'_\beta(1+t^{-\beta/2}).$$

Concerning the estimates involving Λ_x, the proof is exactly the same as the preceding one with Λ_v replaced by Λ_x, β replaced by 3α, $-\partial_v.(\partial_v + v)$ replaced by $-\partial_x.(\partial_x + \partial_x V(x))$ and using the result from [**17**, Proposition 3.1]

$$\left\|(\partial_x + \partial_x V(x))e^{-tK}\right\|_{B\to B} \leq C(1+t^{-3/2}),$$

instead of (2.4). This concludes the proof. □

From Proposition 2.1, it is easy to deduce the following

Corollary 2.3. — *Let $\alpha, \beta \in [0,1]$. Then*

$$\|\Lambda_x^\alpha e^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B\to B} \leq C\big((t-s)^{-1/2+\beta/2-3\alpha/2} + 1\big),$$

and

$$\|\Lambda_v^\beta e^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B\to B} \leq C\big((t-s)^{-1/2} + 1\big).$$

Proof. — We only prove the first statement, the second is similar. By (2.7), (2.8) and also using Remark 2.5 we have

$$\begin{aligned} \|\Lambda_x^\alpha \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B\to B} &\leq \|\Lambda^\alpha \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B\to B} \\ &\leq \|\Lambda^\alpha \mathrm{e}^{-(t-s)K/2}\|_B \|\mathrm{e}^{-(t-s)K/2}\Lambda_v^{1-\beta}\|_B \\ &\leq C\big((t-s)^{-1/2+\beta/2-3\alpha/2} + 1\big), \end{aligned}$$

which was the claim. □

We define

$$B^{\perp} = \Big\{ f \in B \ \ s.t. \ \ \langle f, \mathcal{M}_\infty \rangle = \int f dx dv = 0 \Big\}$$

the orthogonal of $\mathcal{M}_\infty$ in B. At this stage we observe that for $f \in B^\alpha \cap B^\perp$

$$\Lambda_x^\alpha f \in B^\perp, \qquad \Lambda_v^\alpha f \in B^\perp \tag{2.5}$$

and that for all $f \in B^1$

$$\partial_v f \in B^\perp. \tag{2.6}$$

For (2.5) we use that the operator Λ_x is self-adjoint: $\langle \Lambda_x^\alpha f, \mathcal{M}\rangle = \langle f, \Lambda_x^\alpha \mathcal{M}\rangle = 0$ since $\Lambda_x \mathcal{M} = \mathcal{M}$. The same proof holds for Λ_v. The justification of (2.6) is similar using that $\partial_v^* = -(v + \partial_v)$ and $(v + \partial_v)\mathcal{M} = 0$.

A careful analysis shows that we have in fact the following better results when we restrict to $B^\perp$.

Proposition 2.4. — *For all $\alpha, \beta \in [0,1]$ there exist $C_\alpha, C_\beta > 0$ so that for all $t > 0$*

$$\|\Lambda_x^\alpha e^{-tK}\|_{B^\perp \to B^\perp} \leq C_\alpha (1 + t^{-3\alpha/2}) e^{-\kappa t}, \quad \|e^{-tK}\Lambda_x^\alpha\|_{B^\perp \to B^\perp} \leq C_\alpha (1 + t^{-3\alpha/2}) e^{-\kappa t} \tag{2.7}$$

and

$$\|\Lambda_v^\beta e^{-tK}\|_{B^\perp \to B^\perp} \leq C_\beta (1 + t^{-\beta/2}) e^{-\kappa t}, \quad \|e^{-tK}\Lambda_v^\beta\|_{B^\perp \to B^\perp} \leq C_\beta (1 + t^{-\beta/2}) e^{-\kappa t}. \tag{2.8}$$

In the previous bounds, the constants C_α and C_β only depend on a finite number of derivatives of V.

Proof. — For $0 \leq t \leq 1$, this is a direct consequence of the preceding proof and the fact that $B^\perp$ is stable by X_0, Λ_x^2 and Λ_v^2 and therefore Λ^2, K and K^* by direct computations. For $t \geq 1$, the proposition is a consequence of the regularizing properties of e^{-tK} stated in [**18**, Theorem 0.1] and the spectral gap for K: it is proven there that for all $s \in \mathbb{R}$, there exist $N_s > 0$ and $C_s > 0$ such that

$$\forall t > 0, \qquad \big\|\Lambda^s e^{-tK}\Lambda^s\big\|_{B^\perp \to B^\perp} \leq C_s (t^s + t^{-s}) e^{-\kappa t}.$$

Using this and possibly replacing κ by $\kappa/2$ gives the result for $t \geq 1$. This completes the proof. □

Remark 2.5. — In fact possibly replacing once more κ by $\kappa/2$, we also get directly that Proposition 2.4 is also true with K replaced by $K/2$. We shall use this just below.

Similarly to Corollary 2.3 we have the following

Corollary 2.6. — *Let $\alpha, \beta \in [0,1]$. Then*

$$\|\Lambda_x^\alpha e^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp \to B^\perp} \leq C\big((t-s)^{-1/2+\beta/2-3\alpha/2} + 1\big) e^{-\kappa(t-s)}, \tag{2.9}$$

and

$$\|\Lambda_v^\beta e^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp \to B^\perp} \leq C\big((t-s)^{-1/2} + 1\big) e^{-\kappa(t-s)}. \tag{2.10}$$

Proposition 2.7. — *There exists $C > 0$ so that for all $\gamma \in [0,2]$ and all $t \geq 0$*

$$\|\Lambda^\gamma e^{-tK}\Lambda^{-\gamma}\|_{B \to B} \leq C, \tag{2.11}$$

and

$$\|\Lambda^\gamma e^{-tK}\Lambda^{-\gamma}\|_{B^\perp \to B^\perp} \leq C e^{-\kappa t}. \tag{2.12}$$

In the previous bounds, the constant only depends on a finite number of derivatives of V.

Proof. — We only give the proof of (2.11), since (2.12) can be obtained with the same argument. Recall the definition (1.8) of the space $B^{\alpha,\beta}_{x,v}$. We first note that it is equivalent to show that e^{-tK} is bounded from $B^{\gamma,\gamma}_{x,v}$ into itself. We first begin with the case $\gamma = 2$. We therefore look, for an initial data $f_0 \in B^{2,2}_{x,v}$ at the equation satisfied by $g = \Lambda^2 f$ in B. Let us define the operator

$$X_0 = v.\partial_x - \partial_x V.\partial_v.$$

Since

$$\partial_t f + X_0 f - \partial_v.(\partial_v + v)f = 0, \qquad f_{t=0} = f_0$$

and from the regularising properties of e^{-tK}, we get

$$\partial_t g + X_0 g - \partial_v.(\partial_v + v)g = [X_0, \Lambda^2]\Lambda^{-2} g, \qquad g_{t=0} = g_0$$

where we also used that $-\partial_v.(\partial_v + v)$ and Λ^2 commute. Integrating against g in B gives

$$\partial_t \|g\|^2 \leq \left([X_0, \Lambda^2]\Lambda^{-2} g, g\right),$$

since X_0 is skew adjoint and $-\partial_v.(\partial_v + v)$ is non-negative. Let us study the right-hand side commutator. We have

$$\begin{aligned}[X_0, \Lambda_v^2]\Lambda^{-2} &= [v.\partial_x - \partial_x V(x).\partial_v, \Lambda_v^2]\Lambda^{-2} \\ &= \left([v, \Lambda_v^2]\partial_x - [\partial_v, \Lambda_v^2]\partial_x V(x)\right)\Lambda^{-2}.\end{aligned}$$

This gives with a direct computation

$$\left\|[X_0, \Lambda_v^2]\Lambda^{-2} g\right\|_B \leq C \|g\|_B.$$

We can do exactly the same with Λ_x^2 (using that $V^{(3)}$ is bounded) and we get on the whole that

$$\left\|[X_0, \Lambda^2]\Lambda^{-2} g\right\|_B \leq 2C \|g\|_B$$

so that with a new constant $C > 0$

$$\partial_t \|g\|_B^2 \leq 2C \|g\|_B^2.$$

We therefore get

$$\|g(t)\|_B \leq e^{Ct} \|g_0\|_B$$

which we will use for $t \in [0, 1]$. Using the regularising property of e^{-tK} ([**18**, Theorem 0.1]), we also know that for all $t \geq 1$,

$$\|g(t)\|_B \leq \|f(t)\|_{B^2} \leq C' \|f_0\|_{B^2} \leq C' \|g_0\|_B.$$

Putting these results together give for all $t \geq 0$,

$$\|g(t)\|_B \leq C \|g_0\|_B$$

and therefore e^{-tK} is (uniformly in $t > 0$) bounded from B^2 to B^2. Now the result is also clear for $\gamma = 0$ by the semi-group property, and by interpolation we get that e^{-tK} is (uniformly in $t > 0$) bounded from B^γ to B^γ. As a conclusion we get

$$\|\Lambda^\gamma \mathrm{e}^{-tK} \Lambda^{-\gamma}\|_{B \to B} \leq C_\gamma,$$

which was the claim. □

We are now able to state the following interpolation results

Lemma 2.8. — *Let $\beta \in [0, 1]$. Then there exists $C > 0$ so that for all $a \in [0, \beta]$*

$$\|\Lambda_v^\beta e^{-tK}\|_{B^a \to B} \leq C(1 + t^{-(\beta - a)/2}).$$

Proof. — For $a=0$, this follows from Proposition 2.1. Next, set $a=\beta$, then for $f\in B^\beta$

$$\|\Lambda_v^\beta \mathrm{e}^{-tK} f\|_B \leq \|\Lambda^\beta \mathrm{e}^{-tK} f\|_B \leq \|\Lambda^\beta \mathrm{e}^{-tK}\Lambda^{-\beta}\|_{B\to B}\|\Lambda^\beta f\|_B \leq C\|f\|_{B^\beta},$$

by (2.11). The general case $a\in[0,\beta]$ is obtained by interpolation. □

Lemma 2.9. — *Let $0\leq a_0\leq 2$. Then for all $a_0\leq a\leq a_0+2$ there exists $C>0$ so that for all $0\leq t\leq 1$*

$$\|\Lambda^{a_0}\big(e^{-tK}-1\big)\|_{B^a\to B}\leq Ct^{(a-a_0)/2}.$$

Proof. — For $a=a_0$, the result follows from (2.11). Now we prove the bound for $a=a_0+2$, and the general result will follow by interpolation. We write

$$\big(1-\mathrm{e}^{-tK}\big)f=\int_0^t \mathrm{e}^{-sK}Kf ds.$$

Then we use that $K:B^2\longrightarrow B$ is bounded, and by (2.11) we get for all $f\in B^{a_0+2}$

$$\begin{aligned}\|\Lambda^{a_0}\big(\mathrm{e}^{-tK}-1\big)\|_{B^a\to B} &\leq \int_0^t \|\Lambda^{a_0}\mathrm{e}^{-sK}\Lambda^{-a_0}\|\|\Lambda^{a_0}Kf\|ds\\ &\leq Ct\|f\|_{B^{a_0+2}},\end{aligned}$$

hence the result. □

We conclude this section with a technical result.

Lemma 2.10. — *For all $\delta\in\mathbb{R}$ there exists $C_\delta>0$ so that*

(2.13) $$\|\Lambda_v^{-\delta}\Lambda_v^{-1}\partial_v\Lambda_v^\delta\|_{B\to B^\perp}\leq C_\delta.$$

In the previous bound, the constant only depends on a finite number of derivatives of V.

Proof. — From to [**18**, Proposition A.7] we directly get that operator $\Lambda_v^{-\delta}\Lambda_v^{-1}\partial_v\Lambda_v^\delta$ is bounded from B to B. Indeed in the symbolic estimates and pseudo-differential scales introduced there, the operator ∂_v is of order 1 with respect to the velocity variable. Now using the stability of $B^\perp$ by Λ_v and (2.6) yield the result. □

Remark 2.11. — We shall see in the next section (Section 3.6) that most of the results of this section remain true when V is perturbed by a less regular term $\widetilde{V}\in W^{2,\infty}$. We will need this for the small time analysis of the equation (1.1).

3. Intermediate results

In this section, we gather some intermediate results about the Vlasov-(Poisson)-Fokker-Planck equation. In the first subsection we state some a priori basic properties satisfied by solutions of the Fokker-Planck equation and then equation (1.1). In the second one we study more carefully the Poisson term, and in the last one we recall some facts about the equilibrium state.

3.1. The linear Fokker-Planck equation. — In this section, we just recall from [**6**, Appendix A] some standard and basic results about the behaviour of the solutions of the following linear Krammers-Fokker-Planck equation

$$\begin{cases} \partial_t f + v.\partial_x f - (\partial_x V - v).\partial_v f - f - \varepsilon_0 E(t,x)\partial_v f - \Delta_v f = F, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{3.1}$$

Note that equation (1.1) with given field E and $V = V_e$ enters in this setting and that the linear Fokker-Planck equation corresponds to $E = 0$. In both cases we take $F = 0$ and point out that we used the commutation estimate $-\partial_v(\partial_v + v)f = (\partial_v + v)(-\partial_v)f - f$.

For the following, we take $T > 0$ arbitrary and denote by $X = L^2([0,T] \times \mathbb{R}^d_x, H^1_v(\mathbb{R}^d))$ and consider the space $Y = \{f \in X, (\partial_t + v.\partial_x - (\partial_x V - v).\partial_v)f \in X'\}$. The following result is classical and we refer to [**6**, Appendix A] for the proof.

Proposition 3.1. — *Suppose $E \in L^\infty([0,T]\times\mathbb{R}^d)$, $f_0 \in L^2(\mathbb{R}^{2d})$ and $F \in L^2([0,T]\times\mathbb{R}^d_x, H^{-1}_v)$. Then there exists a unique weak solution f of the equation* (3.1) *in the class Y. Moreover*

(i) If $f_0 \geq 0$ then $f \geq 0$.
(ii) If $f_0 \in L^\infty(\mathbb{R}^{2d})$, then for all $0 \leq t \leq T$,

$$\|f(t)\|_{L^\infty(\mathbb{R}^{2d})} \leq e^{dt}\|f_0\|_{L^\infty(\mathbb{R}^{2d})}.$$

This immediately implies the following a priori estimate on the full problem (1.1).

Corollary 3.2. — *Let $f_0 \in L^\infty(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ be such that $f_0 \geq 0$ and consider a solution of* (1.1) *such that the field $E \in L^\infty([0,T] \times \mathbb{R}^d)$. Then, for all $0 \leq t \leq T$, $f(t,.) \geq 0$ and*

$$\|f(t)\|_{L^\infty(\mathbb{R}^{2d})} \leq e^{dt}\|f_0\|_{L^\infty(\mathbb{R}^{2d})}.$$

3.2. Poisson-Emden equation and equilibrium state. — The aim of this subsection is to prove that the potential U_∞ associated to the stationary solutions of the Vlasov-Poisson-Fokker-Planck equation is in $W^{\infty,\infty}(\mathbb{R}^d)$. Recall that the equation satisfied by U_∞ is

$$-\Delta U_\infty = \frac{e^{-(V_e+\varepsilon_0 U_\infty)}}{\int e^{-(V_e+\varepsilon_0 U_\infty)}dx} \tag{3.2}$$

where we recall that ε_0 is varying in a small fixed neighbourhood of 0, and that $\varepsilon_0 > 0$ in the case of dimension $d = 2$.

3.2.1. Case $d = 3$. — When we are in the repulsive interaction case ($\varepsilon_0 > 0$), the existence and uniqueness of a (Green) solution of this equation is given by a result of Dolbeault [**9**] (see also [**10**]) under a light hypothesis on the external potential. We first quote his result in dimension $d = 3$ and in the Coulombian case

Proposition 3.3 **([9], Section 2).** — *Let $U_e \in L^\infty_{loc}(\mathbb{R}^3)$ and $M > 0$. Assume that $e^{-U_e} \in L^1(\mathbb{R}^3)$, then there exists a unique solution $U \in L^{3,\infty}(\mathbb{R}^3)$ of the Poisson-Emden equation*

$$-\Delta U = M\frac{e^{-(U_e+U)}}{\int e^{-(U_e+U)}dx}. \tag{3.3}$$

Moreover $U \geq 0$.

The main property of U which will be needed in the following is $U \geq 0$, that's why we do not even define precisely the space $L^{3,\infty}(\mathbb{R}^3)$. For more details, we address to [**9**].

We then state another result of Bouchut and Dolbeault in the Newtonian case ($\varepsilon_0 < 0$). This result happens to hold only for small M.

Proposition 3.4 ([**3**, Theorem 3.2 and Proposition 3.4]). — *Assume that* $e^{-U_e} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *and is not identically equal to* 0. *Then there exists* $M_0 < 0$ *such that for all* $M_0 < M \leq 0$ *there exists a bounded continuous function of equation* (3.3) *such that* $\lim_{x\to\infty} U(x) = 0$.

Now Assumption 1 on the exterior potential V_e implies that $e^{-V_e} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. As a consequence we can apply Proposition 3.3 at least in the case when ε_0 is small to $U = \varepsilon_0 U_\infty$, $U_e = V_e$, $M = \varepsilon_0$ and $d = 3$ to (3.2) and we get a unique solution U_∞ in $L^{3,\infty}$ when $\varepsilon_0 > 0$. Similarly we can apply Proposition 3.4 when $\varepsilon_0 < 0$ and we get $U_\infty \in L^\infty$. Notice that in our context, $|\varepsilon_0|$ is small and hence both Propositions 3.3 and 3.4 apply here.

Actually, the regularity of U_∞ is improved under the assumption $e^{-V_e} \in \mathcal{S}(\mathbb{R}^3)$, and we can also get some uniformity with respect to the parameter ε_0.

Proposition 3.5. — *Let* $d = 3$. *Suppose that* V_e *satisfies Assumption 1. Then the unique solution* U_∞ *of the Poisson-Emden equation* (3.2) *is in* $W^{\infty,\infty}(\mathbb{R}^3)$, *with semi-norms uniformly bounded w.r.t.* ε_0 *varying in a small fixed neighbourhood of* 0.

Proof of Proposition 3.5. — In order to prove that $U_\infty \in W^{\infty,\infty}$, it is sufficient to prove that the (Green) solution U_∞ of the following Poisson-Emden-type equation

$$-\Delta U_\infty = C_0^{-1} e^{-(V_e+\varepsilon_0 U_\infty)} \tag{3.4}$$

is in $W^{\infty,\infty}$, where

$$C_0 = \int e^{-(V_e(x)+\varepsilon_0 U_\infty(x))} dx$$

is the normalization constant. We first work on $\varepsilon_0 U_\infty$ and note that it is given by

$$\varepsilon_0 U_\infty = \frac{\varepsilon_0 C_0^{-1}}{4\pi} \frac{1}{|x|} \star e^{-(V_e+\varepsilon_0 U_\infty)}.$$

We then consider the Green solution U_e of $-\Delta U_e = e^{-V_e}$ given by

$$U_e = \frac{1}{4\pi|x|} \star e^{-V_e}.$$

From Propositions 3.3 and 3.4 we get directly that $\varepsilon_0 U_\infty$ exists, at least for ε_0 varying in a small neighbourhood of 0, and that it is either non-negative or uniformly bounded. It implies that there exists a constant $C > 0$ uniform in ε_0 such that $0 \leq U_\infty \leq C U_e$ since we also have

$$U_\infty = \frac{C_0^{-1}}{4\pi|x|} \star e^{-(V_e+\varepsilon_0 U_\infty)}.$$

From the Hardy-Littlewood Sobolev inequalities or by a direct computation, we have $U_e \in L^p$ for $3 < p \leq \infty$. Therefore this is also the case for U_∞. Since we directly have that $-\Delta U_\infty \in L^p$ for all $p \in [1, \infty]$ from (3.4), we get that

$$-\Delta U_\infty + U_\infty \in L^p, \quad 3 < p < \infty$$

and this gives $U_\infty \in W^{2,p}$ by elliptic regularity in $\mathbb{R}^d$ (see for example [**28**], [**30**]).

Now we shall use a bootstrap argument to prove that $U_\infty \in W^{\infty,\infty}$. Let $3 < p < \infty$ be fixed in the following. We note that

$$(-\Delta + 1)^2 U_\infty = -\Delta \left(C_0^{-1} e^{-(V_e+\varepsilon_0 U_\infty)} \right) + 2C_0^{-1} e^{-(V_e+\varepsilon_0 U_\infty)} + U_\infty \tag{3.5}$$

and we study each term in order to prove that this expression is uniformly in L^p. Since $U_\infty \in L^\infty$, we have $e^{-\varepsilon_0 U_\infty} \in L^\infty$ and we get for all $1 \leq i, j \leq 3$

$$\partial_{ij}(e^{-\varepsilon_0 U_\infty}) = \left(-\varepsilon_0 \partial_{ij} U_\infty + \varepsilon_0^2 (\partial_i U_\infty)(\partial_j U_\infty)\right) e^{-\varepsilon_0 U_\infty} \in L^p$$

uniformly, since on the one hand $U_\infty \in W^{2,p}$ uniformly and on the other hand

$$\forall k,\quad \partial_k U_\infty \in L^{2p} \Longrightarrow (\partial_i U_\infty)(\partial_j U_\infty) \in L^p. \tag{3.6}$$

In a direct way we also get $\partial_k e^{-\varepsilon_0 U_\infty} \in L^p$. Since $e^{-V_e} \in W^{2,p}$ and using the same trick as in (3.6), this gives from (3.5) that $U_\infty \in W^{4,p}$ for the arbitrary fixed $3 < p < \infty$. By a bootstrap argument using the same method we get that

$$U_\infty \in W^{2k,p},$$

for all $k \in \mathbb{N}$ and therefore

$$U_\infty \in \bigcap_{k\in\mathbb{N}} W^{2k,p} \subset W^{\infty,\infty}.$$

The uniformity w.r.t. ε_0 is also clear and the proof of Proposition 3.5 is complete. □

3.2.2. Case $d = 2$. — We consider here only the Coulombian case ($\varepsilon_0 > 0$).

In this context, we are able to prove the following result

Proposition 3.6. — *Let $d = 2$. Suppose that V_e satisfies Assumption 1. Then the unique solution U_∞ of the Poisson-Emden equation* (3.2) *is in $W^{\infty,\infty}(\mathbb{R}^2)$, with semi-norms uniformly bounded w.r.t. $\varepsilon_0 > 0$ varying in a small fixed neighbourhood of* 0.

Proof. — Notice that when $d = 2$, the equation (3.2) is equivalent to

$$U_\infty = -\frac{1}{2\pi} \ln|x| \star e^{-(V_e+\varepsilon_0 U_\infty)}.$$

The existence and uniqueness of a solution $U_\infty \in L^p(\mathbb{R}^2)$ for any $1 \le p < \infty$ with $\nabla U_\infty \in L^2(\mathbb{R}^2)$ is proved in [**9**, page 199]. Moreover, the maximum principle ensures that $U_\infty \ge 0$. It is then straightforward to adapt the proof of the case $d = 3$ to conclude. □

In the Newtonian case ($\varepsilon_0 < 0$), and for particular choices of V_e (e.g. $V_e(x) = |x|^2$, see [**1**]), there exist solutions to the equation (3.2), but uniqueness is unknown, even under additional assumptions on the solution (radial symmetry, regularity, decay at infinity). However it would be interesting to prove the trend to equilibrium also in this case. We refer to [**1**], where the authors obtained such a result for a related problem.

Remark 3.7. — To end this section we notice that since $U_\infty \in W^{\infty,\infty}(\mathbb{R}^d)$, we get that the potential at infinity $V_e + \varepsilon_0 U_\infty$ satisfies the same hypothesis as V_e alone. As a consequence it will be possible to apply to K_∞ all the properties obtained for any generic Fokker-Planck operator K associated to a generic potential V satisfying Assumptions 1 and 2. This will be crucial in the next section, in which we study the exponential convergence to the equilibrium. A second remark is that the total potential at equilibrium is not explicit. In particular, the Green function for the equation $\partial_t f + K_\infty f$ is not known. This justifies a posteriori the abstract study (anyway with explicit constants) performed in the linear section. In the next section we first go on with the study of a generic linear Fokker-Planck operator by studying the long time behaviour and the exponential decay in time.

3.3. Uniformity of the spectral gap and heat-operator estimates. — The aim of this short subsection is to prove that we have indeed a uniform estimate on the spectral gap for K_∞ with respect to ε_0. Let $d = 2$ or $d = 3$. We work with the operator

$$K_e = v.\partial_x - \partial_x V_e(x).\partial_v - \partial_v.(\partial_v + v)$$

and consider a bound from below κ_0 of the spectral gap of W coming from Assumption 2. From [**18**, Theorem 0.1] we know that there exist constants $C_0, C > 0$ such that for all $t \geq 0$,

$$\left\| e^{-tK_e} \right\|_{B_e^\perp} \leq C_0 e^{-t\kappa_0/C} \tag{3.7}$$

where

$$B_e = \left\{ f \in \mathcal{S}'(\mathbb{R}^{2d}) \ \ s.t. \ \ f\mathcal{M}_e^{-1/2} \in L^2(\mathbb{R}^{2d}) \right\},$$

and $\mathcal{M}_e$ is the Maxwellian associated to V_e and $B_e^\perp$ is the orthogonal of $\mathcal{M}_e$. We then add to the potential a small perturbation of type εU_∞ with $U_\infty \in W^{\infty,\infty}$. This will be applied to the potential U_∞ built in the preceding subsection. Notice that $U_\infty \in W^{\infty,\infty}$ with uniform bounds with respect to $0 < \varepsilon_0 \ll 1$.

The corresponding modified operator is then

$$K_\infty = v.\partial_x - \partial_x V_\infty(x).\partial_v - \partial_v.\left(\partial_v + v\right), \tag{3.8}$$

with $V_\infty = V_e + \varepsilon_0 U_\infty$. The main result is then the following

Proposition 3.8. — *There exists a small real neighbourhood $\mathcal{V}$ of 0 such that for all $t \geq 0$*

$$\left\| e^{-tK_\infty} \right\|_{B^\perp} \leq 4C_0 e^{-t\kappa_0/(8C)}$$

uniformly w.r.t. $\varepsilon_0 \in \mathcal{V}$.

Proof. — We first recall that in (3.7) the precise result of [**18**, Theorem 0.1] says that C_0 depends on a finite number of semi-norms of V_e and that

$$C = \frac{\min\left\{1, \kappa_0\right\}}{64(8 + 3C_e)}$$

where $C_e = \max\left\{ \sup\left\{ \mathrm{Hess}(V_e)^2 - (\frac{1}{4}(\partial_x V_e)^2 - \frac{1}{2}\Delta V_e)\mathrm{Id} \right\}, 0 \right\}$. Adding a small perturbation $\varepsilon_0 U_\infty$ with $U_\infty \in W^{\infty,\infty}$ does only change the constant C into $2C$ and C_0 into $2C_0$ and we only have to check that κ_0 is changed into $\kappa_0/4$ uniformly in ε_0 sufficiently small.

For this we look at the spectrum of

$$W_\infty = -\Delta_x + |\partial_x V_\infty|^2/4 - \Delta_x V_\infty/2$$

and we check that as operators in $L^2(\mathbb{R}^d)$ we have

$$\begin{aligned} W_\infty &= -\Delta_x + |\partial_x V_\infty|^2/4 - \Delta_x V_\infty/2 \\ &\geq -\Delta_x + |\partial_x V_e|^2/4 - \Delta_x V_e/2 + \varepsilon_0^2 |\partial_x U_\infty|^2/4 - |\varepsilon_0||\Delta U_\infty|/2 + \varepsilon_0 \partial_x V_e \partial_x U_\infty/2 \\ &\geq W - \kappa_0/8 + \varepsilon_0 \partial_x V_e \partial_x U_\infty/2 \end{aligned}$$

if we take ε_0 sufficiently small so that $\varepsilon_0^2|\partial_x U_\infty|^2/4 + |\varepsilon_0||\Delta U_\infty|/2 \leq \kappa_0/8$. Now there exist constants a and b such that

$$|\partial_x V_e \partial_x U_\infty| \leq aW + b$$

since V_e has its second order derivatives bounded, and therefore we get for ε_0 sufficiently small

$$W_\infty \geq \frac{1}{2}W - \kappa_0/4.$$

Since $W \geq \kappa_0$, the minmax principle then directly gives that

$$W_\infty \geq \kappa_0/4$$

when restricted to the orthogonal of the 0-eigenspace. The proof is complete. □

Remark 3.9. — We can also notice that the natural norm into the weighted spaces

$$B = \left\{f \in \mathcal{S}' \ \ s.t. \ \ f\mathcal{M}_\infty^{-1/2} \in L^2\right\} \quad \text{and} \quad B_e = \left\{f \in \mathcal{S}' \ \ s.t. \ \ f\mathcal{M}_e^{-1/2} \in L^2\right\}$$

where $\mathcal{M}$ is the Maxwellian associated to V_e, are equivalent with an equivalence constant bounded by $1/2$ uniformly in ε_0 small enough. This justifies the use of the norms associated to the space B instead of the one associated to B_e in the statement of the main theorems of this article.

3.4. Estimates on the Poisson term. — In the following lemma we crucially use the fact that we work in weighted Sobolev spaces instead of flat ones and that $\mathcal{M}_\infty \in \mathcal{S}(\mathbb{R}^{2d})$ uniformly in $|\varepsilon_0| \ll 1$, as proven in the preceding subsection. We have

Lemma 3.10. — *Let $\alpha \in [0,1]$ then there exists $C > 0$ such that for all $h_0 \in B^\alpha$*

$$\Big\| \int h_0 dv \Big\|_{H^\alpha_x} \leq C \left\| h_0 \right\|_{B^\alpha}.$$

Proof. — We work by interpolation. Let us first consider the case $\alpha = 0$. By Cauchy-Schwarz,

$$\begin{aligned}
\Big\| \int h_0 dv \Big\|_{L^2_x} &= \Big\| \int h_0 \mathcal{M}_\infty^{-1/2} \mathcal{M}_\infty^{1/2} dv \Big\|_{L^2_x} \\
&\leq \Big\| \Big(\int h_0^2 \mathcal{M}_\infty^{-1} dv \Big)^{1/2} \big(\int \mathcal{M}_\infty dv \big)^{1/2} \Big\|_{L^2_x} \leq C_0 \left\| h_0 \right\|_B.
\end{aligned}$$

Now we consider the case $\alpha = 1$. We write

$$\begin{aligned}
\Big\| \partial_x \int h_0 dv \Big\|_{L^2_x} &= \Big\| \int \partial_x h_0 dv \Big\|_{L^2_x} \\
&\leq \Big\| \int (\partial_x + \partial_x V) h_0 dv \Big\|_{L^2_x} + \Big\| \int (\partial_x V) h_0 dv \Big\|_{L^2_x} \\
&= \Big\| \int \big((\partial_x + \partial_x V) h_0\big) \mathcal{M}_\infty^{-1/2} \mathcal{M}_\infty^{1/2} dv \Big\|_{L^2_x} + \Big\| \int h_0 \mathcal{M}_\infty^{-1/2} \big(\partial_x V \mathcal{M}_\infty^{1/2}\big) dv \Big\|_{L^2_x} \\
&\leq \Big\| \Big(\int \big((\partial_x + \partial_x V) h_0\big)^2 \mathcal{M}_\infty^{-1} dv \Big)^{1/2} \big(\int \mathcal{M}_\infty dv \big)^{1/2} \Big\|_{L^2_x} \\
&\qquad + \Big\| \Big(\int h_0^2 \mathcal{M}_\infty^{-1} dv \Big)^{1/2} \Big(\int (\partial_x V)^2 \mathcal{M}_\infty dv \Big)^{1/2} \Big\|_{L^2_x} \\
&\leq C \Big\| (\partial_x + \partial_x V) h_0 \Big\|_B + C \left\| h_0 \right\|_B \leq C \left\| h_0 \right\|_{B^1}
\end{aligned}$$

where we used that $(\partial_x V)^2 \mathcal{M}_\infty \in L^\infty_x L^2_v$, and that

$$\begin{aligned}
\| (\partial_x + \partial_x V) h_0 \|_B^2 + \| h_0 \|_B^2 &= (-\partial_x (\partial_x + \partial_x V) h_0, h_0)_B + \| h_0 \|_B^2 \\
&= (\Lambda_x^2 h_0, h_0)_B = \| \Lambda_x h_0 \|_B^2 \leq \| h_0 \|_{B^1}^2.
\end{aligned}$$

This gives the result for $\alpha = 1$. The complete result follows by interpolation. □

Lemma 3.11. — *Assume that $d = 2$ or $d = 3$. Let $h_0 \in B$ and denote by*

$$E_0(x) = \frac{x}{|x|^d} \star \int h_0(x, v) dv.$$

(i) Case $d = 2$. For all $0 < \varepsilon \leq 1/2$ there exists $C > 0$ so that

$$\| E_0 \|_{L^\infty(\mathbb{R}^2)} \leq C \| h_0 \|_{B^\varepsilon}. \tag{3.9}$$

(ii) Case $d = 3$. For all $0 < \varepsilon \leq 1/2$ there exists $C > 0$ so that

$$\| E_0 \|_{L^\infty(\mathbb{R}^3)} \leq C \| h_0 \|_{B^{1/2+\varepsilon}}. \tag{3.10}$$

Proof. — Let us first recall the Hardy-Littlewood-Sobolev inequality (see e.g. [**22**]) which will be useful in the sequel. For all $1 < p, q < +\infty$ such that $\frac{1}{q} - \frac{1}{p} + \frac{1}{d} = 0$

$$\big\| \frac{x}{|x|^d} \star f \big\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.11}$$

We prove (ii). We consider the Fourier multiplier $L_x = (1 - \Delta_x)^{1/2}$. Then, by Hardy-Littlewood-Sobolev and the Sobolev embeddings, for any $\varepsilon > 0$

$$\|E_0\|_{L^\infty(\mathbb{R}^3)} \leq C \big\| L_x^\varepsilon \int h_0 dv \big\|_{L^3(\mathbb{R}^3)} \leq C \big\| L_x^{1/2+\varepsilon} \int h_0 dv \big\|_{L^2(\mathbb{R}^3)} \leq \big\| \int h_0 dv \big\|_{H^{1/2+\varepsilon}(\mathbb{R}^3)}.$$

Using Lemma 3.10 with $\alpha = 1/2 + \varepsilon$ we get (3.10).

The proof of (i) is analogous with L_x^ε replaced with $L_x^{1/2+\varepsilon}$. □

Corollary 3.12. — *Assume that $d = 2$ or $d = 3$. Let $f_0 \in B^\perp$ and denote by*

$$E_0(t,x) = \frac{x}{|x|^d} \star \int e^{-tK} f_0(x,v) dv.$$

(i) Case $d = 2$. For all $0 < \varepsilon \leq 1/2$ there exists $C > 0$ so that for all $t > 0$

$$\|E_0(t,.)\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + t^{-3\varepsilon/2}) e^{-\kappa t} \|f_0\|_B. \tag{3.12}$$

(ii) Case $d = 3$. For all $0 < \varepsilon \leq 1/2$ there exists $C > 0$ so that for all $t > 0$

$$\|E_0(t,.)\|_{L^\infty(\mathbb{R}^3)} \leq C e^{-\kappa t} \|f_0\|_{B^{1/2+\varepsilon}}. \tag{3.13}$$

Proof. — (i). We apply the result of Lemma 3.11 to the case $h_0 = \mathrm{e}^{-tK} f_0$ for some $f_0 \in B^\perp$, then

$$\|E_0(t,.)\|_{L^\infty(\mathbb{R}^2)} \leq C \|\Lambda_x^\varepsilon \mathrm{e}^{-tK} f_0\|_B. \tag{3.14}$$

Thus estimate (2.7) together with (3.14) implies

$$\|E_0(t,.)\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + t^{-3\varepsilon/2}) e^{-\kappa t} \|f_0\|_B,$$

which was to prove.

(ii). By (3.10) and (2.11), we obtain

$$\|E_0(t,.)\|_{L^\infty(\mathbb{R}^3)} \leq C \|\Lambda^{1/2+\varepsilon} \mathrm{e}^{-tK} \Lambda^{-1/2-\varepsilon}\|_{B^\perp \to B^\perp} \|\Lambda^{1/2+\varepsilon} f_0\|_B \leq C e^{-\kappa t} \|f_0\|_{B^{1/2+\varepsilon}},$$

which was the claim. □

3.5. Integral estimates.

— In this subsection we give a technical result.

Lemma 3.13. — *Let $\gamma_1, \gamma_2, c > 0$ and assume that $\gamma_1 \leq 1$. Then there exists $C > 0$ so that for all $t > 0$*

$$\int_0^t \big(s^{-1+\gamma_1} + 1\big)\big((t-s)^{-1+\gamma_2} + 1\big) e^{-c(t-s)} ds \leq \begin{cases} C\big(t^{-1+\gamma_1+\gamma_2} + 1\big) & \text{for} \quad t \leq 1, \\ C & \text{for} \quad t \geq 1. \end{cases} \tag{3.15}$$

Proof. — The proof is elementary: we expand the r.h.s. of (3.15) and estimate each piece. Let $t \leq 1$, then

$$\int_0^t \big(s^{-1+\gamma_1} + 1\big)\big((t-s)^{-1+\gamma_2} + 1\big) \mathrm{e}^{-c(t-s)} ds \leq \int_0^t \big(s^{-1+\gamma_1} + 1\big)\big((t-s)^{-1+\gamma_2} + 1\big) ds,$$

Firstly,

$$\int_0^t s^{-1+\gamma_1} (t-s)^{-1+\gamma_2} ds = C_{\gamma_1,\gamma_2} t^{-1+\gamma_1+\gamma_2},$$

by a simple change of variables. Then for $t \leq 1$

$$\int_0^t s^{-1+\gamma_1} ds + \int_0^t (t-s)^{-1+\gamma_2} ds \leq C,$$

and this yields the result. Now we assume that $t \geq 1$. Then on the one hand

$$\begin{aligned} \int_0^1 \big(s^{-1+\gamma_1}+1\big)\big((t-s)^{-1+\gamma_2}+1\big)\mathrm{e}^{-c(t-s)} ds &\leq C\mathrm{e}^{-ct} \int_0^1 \big(s^{-1+\gamma_1}+1\big)\big((t-s)^{-1+\gamma_2}+1\big) ds \\ &\leq C, \end{aligned}$$

and on the other hand, since $\gamma_1 \leq 1$

$$\begin{aligned} \int_1^t \big(s^{-1+\gamma_1}+1\big)\big((t-s)^{-1+\gamma_2}+1\big)\mathrm{e}^{-c(t-s)} ds &\leq C \int_0^t \big((t-s)^{-1+\gamma_2}+1\big)\mathrm{e}^{-c(t-s)} ds \\ &\leq C, \end{aligned}$$

which completes the proof. □

3.6. Low regularity heat estimates. — In this subsection we show how some of the previous results on the Fokker-Planck operator with potential satisfying Assumption 1 remain valid when the potential is of type

$$V = V_e + \varepsilon_0 U_0$$

where V_e satisfies Assumption 1, $U_0 \in W^{2,\infty}$ and $|\varepsilon_0| \leq 1$. This will be applied in Section 5 when the study for short time will be done.

In the following we denote by

$$K = v.\partial_x - \partial_x V_e(x).\partial_v - \partial_v.\left(\partial_v + v\right)$$

and

$$K_0 = K - \varepsilon_0 \partial_x U_0(x).\partial_v.$$

Note that the Hilbert spaces of type B defined in (1.5) with either $\mathcal{M}_\infty$ (defined in (1.2)) or $\mathcal{M}_e$ (when $V_e + \varepsilon_0 U_\infty$ is replaced there by V_e only) or even $\mathcal{M}_0$ (when $V_e + \varepsilon_0 U_\infty$ is replaced there by $V_e + \varepsilon_0 U_0$) are all equal with equivalent norms uniformly in $0 \leq \varepsilon_0 \leq 1$ and depending only on the norm sup of U_0 or U_∞.

We will need the following result

Lemma 3.14. — *The domains of K and K_0 coincide, they are both maximal accretive with $\mathcal{M}^{1/2}\mathcal{S}$ as a core.*

Proof. — This is clear for K as already noticed and used (see [**18**]). The difficulty is that K_0 has only $W^{1,\infty}$ coefficients. There exists $C_0 > 0$ such that $\|\partial_x U_0\|_{L^\infty} \leq C_0$, and then for any $\eta > 0$, there exists $C_\eta > 0$ such that

$$\|\partial_x U_0.\partial_v f\|_B \leq C_0 \left\|\partial_v f\right\|_B \leq \eta \left\|K f\right\|_B + C_\eta \left\|f\right\|_B,$$

which directly implies that the domains are the same, see e.g. [**11**, Chapter III, Lemma 2.4]. The fact that $\mathcal{M}^{1/2}\mathcal{S}$ is a core is also a direct consequence of this inequality. □

We now prove that some results from Section 2 about semigroup estimates remain true for the new operator K_0 with non-smooth coefficients.

We begin with a general Proposition

Proposition 3.15. — *Let us consider the operator K_0 with potential $V_e + \varepsilon_0 U_0$. Then there exists $C_0 > 0$ such that the following is true uniformly in $\varepsilon_0 \in [0, 1]$ and $t \in (0, 1]$*

(i) $\forall\gamma\in[0,1]$, $\quad \left\|\Lambda^\gamma e^{-tK_0}\Lambda^{-\gamma}\right\|_{B\to B}\le C_0$,

(ii) $\forall\beta\in[0,1]$, $\quad \left\|\Lambda_v^\beta e^{-tK_0}\right\|_{B\to B}\le C_0 t^{-\beta/2}$,

(iii) $\forall\alpha\in[0,1]$, $\quad \left\|\Lambda_x^\alpha e^{-tK_0}\right\|_{B\to B}\le C_0 t^{-3\alpha/2}$,

(iv) $\forall a\in[0,2]$ *and* $f\in B^a$, $\quad \left\|(e^{-tK_0}-1)f\right\|_B\le C_0 t^a\left\|f\right\|_{B^a}$.

Proof. — We first note that the proof of point (iv) given in Lemma 2.9 is unchanged (for $a_0=0$) under the new assumptions on the potential V, and uniformly w.r.t. ε_0. For points (iii) and (ii) this is the same w.r.t. the proof of Proposition 2.1 and we emphasise that the constants only depend on the second derivatives of the potential, which are here uniformly bounded w.r.t. ε_0.

It therefore only remains to check point (i) for which the proof of point (2.11) cannot be directly adapted, since we have to restrict here to the case when $\gamma\in[0,1]$. We have to show that e^{-tK} is bounded from $B^{\gamma,\gamma}_{x,v}$ into itself. We first begin with the case $\gamma=1$. We now use that

$$\|f\|_{B^1}\sim\|\Lambda f\|_B\sim\|(\partial_x+\partial_x V)f\|_B+\|(\partial_v+v)f\|_B$$

with uniform w.r.t. ε_0 equivalence constants, since $U_0\in W^{2,\infty}$. We therefore look, for an initial data $f_0\in B^{1,1}_{x,v}$ at the equation satisfied by $g=(\partial_x+\partial_x V)f$ and $h=(\partial_v+v)f$ in B. We consider again the operator $X_0=v.\partial_x-\partial_x V_e.\partial_v$. Since

$$\partial_t f+X_0 f-\varepsilon_0\partial_x U_0.\partial_v f-\partial_v.(\partial_v+v)f=0,\qquad f_{t=0}=f_0$$

we get the system

$$\begin{aligned}
&\partial_t g+X_0 g-\varepsilon_0\partial_x U_0\partial_v g-\partial_v.(\partial_v+v)g=\mathrm{Hess}V h\\
&\partial_t h+X_0 h-\varepsilon_0\partial_x U_0\partial_v h-\partial_v.(\partial_v+v)h=-h-g+\varepsilon_0\partial_x U_0 f,\\
&\text{with}\qquad g_{t=0}=g_0\in B\quad\text{and}\quad h_{t=0}=h_0\in B.
\end{aligned}$$

Integrating the three last equations against respectively f, g and h in B gives,

$$\partial_t(\|f\|_B^2+\|g\|_B^2+\|h\|_B^2)\le C(\|f\|_B^2+\|g\|_B^2+\|h\|_B^2)$$

since V has a Hessian uniformly bounded w.r.t. ε_0. We therefore get

$$\|f(t)\|_B+\|g(t)\|_B+\|h(t)\|_B\le C_1e^{C_2t}(\|f_0\|_B+\|g_0\|_B+\|h_0\|_B)$$

and we get that e^{-tK_0} is (uniformly in $t\in[0,1]$ and $\varepsilon_0\in[0,1]$) bounded from B^1 to B^1. Now the result is also clear for $\gamma=0$ by the semi group property, and by interpolation we get that e^{-tK_0} is (uniformly in $t\in[0,1]$ and $\varepsilon_0\in[0,1]$) bounded from B^γ to B^γ for $\gamma\in[0,1]$. As a conclusion we get

$$\|\Lambda^\gamma \mathrm{e}^{-tK}\Lambda^{-\gamma}\|_{B\to B}\le C_\gamma.$$

This concludes the proof of point (i) and the proof of the Proposition. □

As a consequence, a certain number of results of Section 2 remain true with proofs without changes. We gather them in the following corollary.

Corollary 3.16. — *There exists $C>0$ such that the following is true uniformly in $\varepsilon_0\in[0,1]$ and $t\in(0,1)$*

(i) $\forall\beta\in[0,1]$, $\forall a\in[0,\beta]$, $\quad \left\|\Lambda_v^\beta e^{-tK_0}\right\|_{B^a\to B}\le C(1+t^{-(\beta-a)/2})$,

(ii) $\forall\beta\in[0,1]$, $\quad \left\|\Lambda_v^\beta e^{-tK_0}\Lambda_v^{1-\beta}\right\|_{B\to B}\le C(1+t^{-1/2})$,

(iii) $\forall\alpha,\beta\in[0,1]$, $\left\|\Lambda^\alpha e^{-tK_0}\Lambda_v^{1-\beta}\right\|_{B\to B}\le C(1+t^{-1/2+\beta/2-3\alpha/2})$.

Proof. — The proof of (*i*) follows the one of Lemma 2.8 thanks to points (*i*), (*ii*) and (*iii*) in Proposition 3.15. Points (*ii*) and (*iii*) are consequences respectively of (*ii*) and (*iii*) of Proposition 3.15 since

$$\left\|\Lambda_v^{\beta} e^{-tK_0}\Lambda_v^{1-\beta}\right\|_{B\to B} \le \left\|\Lambda_v^{\beta} e^{-tK_0/2}\right\|_{B\to B}\left\|e^{-tK_0/2}\Lambda_v^{1-\beta}\right\|_{B\to B}$$

and

$$\left\|\Lambda^{\alpha} e^{-tK_0}\Lambda_v^{1-\beta}\right\|_{B\to B} \le \left\|\Lambda^{\alpha} e^{-tK_0/2}\right\|_{B\to B}\left\|e^{-tK_0/2}\Lambda_v^{1-\beta}\right\|_{B\to B}.$$

□

Remark 3.17. — Let us observe that if one only has $f_0 \in B(\mathbb{R}^6)\cap L^\infty(\mathbb{R}^6)$, one can prove that U_0 defined in (1.10) satisfies $U_0 \in W^{2,p}(\mathbb{R}^3)$ for any $2 \le p < \infty$. In other words, the assumption $U_0 \in W^{2,\infty}(\mathbb{R}^3)$ fills in an $\varepsilon-$gap of regularity. More precisely, let $p \ge 2$ and $q \le 2$ such that $1/p + 1/q = 1$. Then, by Hölder

$$|\Delta U_0| = \int f_0 dv \le \Big(\int f_0^p \mathcal{M}^{-1} dv\Big)^{1/p}\Big(\int \mathcal{M}^{q/p} dv\Big)^{1/q}.$$

Thus using that $\int \mathcal{M}^{q/p} dv \in L^\infty(\mathbb{R}^3)$, we get

$$\int |\Delta U_0|^p dx \le C\int f_0^p \mathcal{M}^{-1} dv dx \le C\|f_0\|^{p-2}_{L^\infty(\mathbb{R}^6)}\|f_0\|_B^2,$$

which implies that $U_0 \in W^{2,p}(\mathbb{R}^3)$ by elliptic regularity.

Now we prove a result that will be useful for the short time analysis in the next section. Again we work with the linear Fokker-Planck operator K_0 with potential $V_e + \varepsilon_0 U_0$.

Lemma 3.18. — *Assume that $d = 3$ and $a > 1/2$. Let $f_0 \in B^a(\mathbb{R}^6)\cap L^\infty(\mathbb{R}^6)$ and denote by*

$$S_0(t,x) = \frac{x}{|x|^3}\star\int\big(e^{-tK_0}-1\big) f_0(x,v)dv.$$

Then for all $\varepsilon \ll 1$ and $0 \le t \le 1$ and uniformly in $\varepsilon_0 \in [0,1]$ we have

$$\|S_0(t)\|_{L^\infty(\mathbb{R}^3)} \le C t^{a/3-\varepsilon}\big(\|f_0\|_{L^\infty} + \|f_0\|_{B^a}\big). \tag{3.16}$$

Proof. — In the sequel, $0 \le t \le 1$ is fixed. Let $\sigma = a - 1/2 > 0$ and let $q > 3/\sigma$ be large. Then by the Gagliardo-Nirenberg inequality

$$\|S_0\|_{L_x^\infty} \le C\|S_0\|^{1-\frac{3}{\sigma q}}_{L_x^q}\|S_0\|^{\frac{3}{\sigma q}}_{W_x^{\sigma,q}}, \tag{3.17}$$

and we now estimate the previous terms.

By (3.11), there exists $p < 3$ (with $p \longrightarrow 3$ when $q \longrightarrow +\infty$) such that

$$\|S_0\|_{L_x^q} \le C\Big\|\int h_0 dv\Big\|_{L_x^p},$$

where $h_0 = \big(\mathrm{e}^{-tK_0} - 1\big) f_0$. Then, by Hölder (where p' is the conjugate of p)

$$\begin{aligned}\int |h_0| dv &= \int \big(|h_0|\mathcal{M}_\infty^{-1/p}\big)\mathcal{M}_\infty^{1/p} dv\\ &\le \Big(\int |h_0|^p\mathcal{M}_\infty^{-1}dv\Big)^{1/p}\Big(\int \mathcal{M}_\infty^{p'/p}dv\Big)^{1/p'}\\ &\le C\Big(\int |h_0|^p\mathcal{M}_\infty^{-1}dv\Big)^{1/p}.\end{aligned}$$

This implies that

$$\|S_0\|_{L_x^q} \le C\Big\|\int h_0 dv\Big\|_{L_x^p} \le C\Big(\int |h_0|^p\mathcal{M}_\infty^{-1}dvdx\Big)^{1/p} \le C\|h_0\|^{1-2/p}_{L^\infty}\|h_0\|_B^{2/p}. \tag{3.18}$$

Now, by point (iv) of Proposition 3.15 we have $\|h_0\|_{L^\infty} \leq C\|f_0\|_{L^\infty}$, and by Lemma 2.9, $\|h_0\|_B \leq Ct^{a/2}\|f_0\|_{B^a}$, hence

$$\|S_0\|_{L^q_x} \leq Ct^{a/p}\big(\|f_0\|_{L^\infty} + \|f_0\|_{B^a}\big).$$

Next, by (3.11) and Sobolev (recall that $p \sim 3$ for q large)

$$\|S_0\|_{W^{\sigma,q}_x} \leq C\big\|(1-\Delta_x)^{\sigma/2}\int h_0 dv\big\|_{L^p_x} \leq C\big\|(1-\Delta_x)^{a/2}\int h_0 dv\big\|_{L^2_x},$$

since $\sigma + 1/2 = a$. Then we proceed as in the proof of (3.10) to get

$$\|S_0\|_{W^{\sigma,q}_x} \leq C\|h_0\|_{B^a} \leq C\|f_0\|_{B^a}. \tag{3.19}$$

Fix $\varepsilon \ll 1$. Then for $q \gg 1$, we combine (3.17), (3.18) and (3.19) to get (3.16). □

4. Proof of Theorem 1.2 (case $d = 2$)

4.1. Functional setting. — To begin with, we introduce the functional framework which will be used in both cases $d = 2$ or $d = 3$.

To show the trend to equilibrium, we look for a solution of the form $f = f_\infty + g$ with $f_\infty = c\mathcal{M}_\infty$ and $g \in B^\perp$. The normalization $\int f dxdv = \int \mathcal{M}_\infty dxdv = 1$ then implies that $f_\infty = \mathcal{M}_\infty$. Hence we write

$$f = \mathcal{M}_\infty + g, \qquad E = E_\infty + F,$$

with

$$\partial_x U_\infty = E_\infty = -\frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star \int \mathcal{M}_\infty dv, \qquad F = -\frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star \int g dv.$$

In the sequel denote by

$$K = K_\infty.$$

We want to take profit of the regularization property stated in Lemma 3.11, thus we look for a solution of the form

$$g = \mathrm{e}^{-tK} g_0 + h, \qquad F = F_0 + G,$$

with

$$F_0 = -\frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star \int \mathrm{e}^{-tK} g_0 dv, \qquad G = -\frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star \int h dv,$$

and $h(0) = G(0) = 0$. At this stage we observe that $f_0 = \mathcal{M}_\infty + g_0$ and that for all $t \geq 0$, $\mathrm{e}^{-tK} f_0 = \mathcal{M}_\infty + \mathrm{e}^{-tK} g_0$.

We construct the solution with a fixed point argument on (h, G), and therefore we define the map $\Phi = (\Phi_1, \Phi_2)$ given by

$$\Phi_1(h,G)(t) = \varepsilon_0 \int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s) + G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big) ds$$

$$\Phi_2(h,G)(t) = -\frac{\varepsilon_0}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} \star \int_{\mathbb{R}^d}\int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s) + G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big) dsdv,$$

and we observe that (f, E) solves (1.9) if and only if $(h, G) = \Phi(h, G)$. For $\alpha, \beta, \gamma, \delta, \sigma \geq 0$ define the norms

$$\|h\|_{X^{\alpha,\beta}_\delta} = \sup_{t\geq 0}\Big(\frac{t^\delta}{1+t^\delta} e^{\sigma\kappa t}\|h(t,.)\|_{B^{\alpha,\beta}_{x,v}(\mathbb{R}^{2d})}\Big),$$

$$\|G\|_Y = \sup_{t\geq 0}\Big(e^{\sigma\kappa t}\|G(t,.)\|_{L^\infty(\mathbb{R}^d)}\Big),$$

define the Banach space

$$Z := X^{\alpha,\beta}_\delta \times Y, \qquad \text{with} \qquad \|(h,G)\|_Z = \max\big(\|h\|_{X^{\alpha,\beta}_\delta}, \|G\|_Y\big),$$

and denote by Γ_1 its unit ball. In each of the cases $d=2$ or 3, for a given initial condition g_0, we will prove that if $|\varepsilon_0|<1$ is small enough, the map Φ is a contraction of the ball $\Gamma_1 \subset Z$. To alleviate notations, we assume in the sequel that $\varepsilon_0>0$.

4.2. The fixed point argument in the case $d=2$. — This case is the easiest. Let $g_0 \in B$. We can fix here $\alpha=\beta=\delta=0$. Let $\varepsilon \ll 1$ and fix $\sigma = 1/2$. For simplicity, we write $X = X^{0,0}_0$.

We proceed in two steps. Recall that Γ_1 is the unit ball of Z. Then

Step1: Φ maps the ball $\Gamma_1 \subset Z$ into itself

• We estimate $\Phi_1(h,G)$ in X. By (2.13) and (2.6), we have for all $t \geq 0$

$$\begin{aligned}\|\Phi_1(h,G)(t)\|_B &\leq \varepsilon_0 \int_0^t \|\mathrm{e}^{-(t-s)K}\big(F_0+G\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\big)\|_B\, ds \\ &\leq C\varepsilon_0 \int_0^t \|F_0+G\|_{L^\infty(\mathbb{R}^2)}\|\mathrm{e}^{-(t-s)K}\Lambda_v\|_{B^\perp}\|\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\|_B\, ds, \qquad (4.1)\end{aligned}$$

and we estimate each factor in the previous integral.

<u>Estimation of $\|\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\|_B$</u>: We use that $\mathcal{M}_\infty \in B$, and by (2.1) we obtain

$$\begin{aligned}\|\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\|_B &\leq \|\mathcal{M}_\infty\|_B + \|\mathrm{e}^{-sK}g_0\|_B + \|h(s)\|_B \\ &\leq C(1+\|h\|_X). \qquad (4.2)\end{aligned}$$

<u>Estimation of $\|F_0+G\|_{L^\infty(\mathbb{R}^3)}$</u>: By (3.12) we get

$$\begin{aligned}\|F_0+G\|_{L^\infty(\mathbb{R}^2)} &\leq \|F_0\|_{L^\infty(\mathbb{R}^2)} + \|G\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C(1+s^{-3\varepsilon/2})e^{-\sigma\kappa s}\|g_0\|_B + C\mathrm{e}^{-\sigma\kappa s}\|G\|_Y \\ &\leq C(1+s^{-3\varepsilon/2})\mathrm{e}^{-\sigma\kappa s}(1+\|G\|_Y). \qquad (4.3)\end{aligned}$$

<u>Estimation of $\|\mathrm{e}^{-(t-s)K}\Lambda_v\|_{B^\perp\to B^\perp}$</u>: This follows from (2.8)

$$\|\mathrm{e}^{-(t-s)K}\Lambda_v\|_{B^\perp\to B^\perp} \leq C\big((t-s)^{-1/2}+1\big)e^{-\kappa(t-s)}. \qquad (4.4)$$

Therefore by (4.1), (4.2), (4.3) and (4.4) we have

$$\|\Phi_1(h,G)(t)\|_B \leq C\varepsilon_0(1+\|h\|_X)(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\int_0^t \big(s^{-3\varepsilon/2}+1\big)\big((t-s)^{-1/2}+1\big)e^{-\kappa(t-s)/2}ds.$$

Now, by (3.15) we deduce

$$\|\Phi_1(h,G)(t)\|_B \leq C\varepsilon_0(1+\|h\|_X)(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t},$$

which in turn yields the bound

$$\|\Phi_1(h,G)\|_X \leq C\varepsilon_0(1+\|h\|_X)(1+\|G\|_Y) \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2. \qquad (4.5)$$

• We turn to the estimation of $\|\Phi_2(h,G)\|_Y$. We apply (3.9) with

$$h_0 = \int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s)+G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\big)ds,$$

then for all $t \geq 0$

$$\begin{aligned}\|\Phi_2(h,G)(t)\|_{L^\infty(\mathbb{R}^2)} &\leq C\|\Lambda^\varepsilon \int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s)+G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\big)ds\|_B \\ &\leq C\int_0^t \|\Lambda^\varepsilon \mathrm{e}^{-(t-s)K}\big(F_0+G\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\big)\|_B ds \\ &\leq C\varepsilon_0 \int_0^t \|F_0+G\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^\varepsilon \mathrm{e}^{-(t-s)K}\Lambda_v\|_{B^\perp}\|\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\|_B ds,\end{aligned}$$

where in the last line we used (2.13). Then by (4.2), (4.3) and (2.9) with $\alpha = \varepsilon$ and $\beta = 0$ we get

$$\|\Phi_2(h,G)(t)\|_{L^\infty(\mathbb{R}^2)} \leq$$
$$\leq C\varepsilon_0(1+\|h\|_X)(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\int_0^t \big(s^{-3\varepsilon/2}+1\big)\big((t-s)^{-1/2-3\varepsilon/2}+1\big)e^{-\kappa(t-s)/2}ds.$$

By (3.15), this in turn implies

$$\|\Phi_2(h,G)\|_Y \leq C\varepsilon_0(1+\|h\|_X)(1+\|G\|_Y) \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2. \tag{4.6}$$

As a result, by (4.5) and (4.6) there exists $C>0$ such that

$$\|\Phi(h,G)\|_Z \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2.$$

Therefore we can choose $\varepsilon_0 > 0$ small enough so that Φ maps the ball $\Gamma_1 \subset Z$ into itself.

Step2: Φ is a contraction of Γ_1

With exactly the same arguments, we can also prove the contraction estimate

$$\|\Phi_1(h_2,G_2)-\Phi_1(h_1,G_2)\|_Z \leq C\varepsilon_0\|(h_2-h_1,G_2-G_1)\|_Z\big(1+\|(h_1,G_1)\|_Z+\|(h_2,G_2)\|_Z\big).$$

We do not write the details.

As a conclusion, if $\varepsilon_0 > 0$ is small enough, Φ has a unique fixed point in $\Gamma_1 \subset Z$. This shows the existence of a unique $h \in \mathcal{C}\big([0,+\infty[\,;\,B(\mathbb{R}^4)\big)$ such that $f = \mathcal{M}_\infty + \mathrm{e}^{-tK}g_0 + h$ solves (1.1).

4.3. Conclusion of the proof of Theorem 1.2. — The convergence of f to equilibrium follows from the choice of the weighted spaces. By definition

$$\|h(t)\|_B \leq C\mathrm{e}^{-\sigma\kappa t}\|h\|_X \longrightarrow 0, \quad \text{when} \quad t \longrightarrow +\infty.$$

Similarly,

$$\|G(t)\|_{L^\infty} \leq C\mathrm{e}^{-\sigma\kappa t}\|G\|_Y \longrightarrow 0, \quad \text{when} \quad t \longrightarrow +\infty.$$

5. Proof of Theorem 1.3 (case $d=3$)

5.1. Small time analysis: $0 \leq t \leq 1$. — To begin with we prove a local well-posedness result for (1.1).

Proposition 5.1. — *Let $d=3$ and $1/2 < a < 3/4$. Assume that $f_0 \in B^{a,a}(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ is such that U_0 defined in* (1.10) *is in $W^{2,\infty}(\mathbb{R}^3)$. Assume moreover that Assumptions 1 and 2 are satisfied. Then if $|\varepsilon_0|$ is small enough, there exists a unique local mild solution f to* (1.1) *in the class*

$$f \in \mathcal{C}\big([0,1]\,;\,B^{a,a}(\mathbb{R}^6)\big) \cap L^\infty\big([0,1]\,;\,L^\infty(\mathbb{R}^6)\big).$$

We write

$$f = \mathrm{e}^{-tK_0} f_0 + g, \qquad E = E_0 + F,$$

where U_0, E_0 and F are defined by

$$\partial_x U_0 = E_0 = -\frac{1}{|\mathbb{S}^2|}\frac{x}{|x|^3} \star \int f_0 dv, \qquad F = -\frac{1}{|\mathbb{S}^2|}\frac{x}{|x|^3} \star \int g dv.$$

In the regime $0 \le t \le 1$, the effective Fokker-Planck operator is given by

$$K_0 = v.\partial_x - \partial_x V_0(x).\partial_v - \partial_v.\left(\partial_v + v\right),$$

where $V_0 = V_e + \varepsilon_0 U_0$. The mild formulation of (1.1), using K_0, is therefore

$$\begin{cases} f(t) = e^{-tK_0} f_0 + \varepsilon_0 \displaystyle\int_0^t e^{-(t-s)K_0}(E(s) - E_0)\partial_v f(s) ds, \\ E(t) = -\dfrac{1}{|\mathbb{S}^2|}\dfrac{x}{|x|^3} \star_x \displaystyle\int f(t) dv. \end{cases} \tag{5.1}$$

We construct the solution with a fixed point argument on (g, F), and therefore we define the map $\Phi = (\Phi_1, \Phi_2)$ given by

$$\Phi_1(g, F)(t) = \varepsilon_0 \int_0^t \mathrm{e}^{-(t-s)K_0} F(s)\partial_v\big(\mathrm{e}^{-sK_0} f_0 + g(s)\big) ds$$

$$\Phi_2(g, F)(t) = -\frac{1}{|\mathbb{S}^2|}\frac{x}{|x|^3} \star \int_{\mathbb{R}^3} \Big[\big(e^{-tK_0} - 1\big) f_0 + \varepsilon_0 \int_0^t \mathrm{e}^{-(t-s)K_0} F(s)\partial_v\big(\mathrm{e}^{-sK_0} f_0 + g(s)\big) ds\Big],$$

and we observe that (f, E) solves (5.1) if and only if $(g, F) = \Phi(g, F)$. For $\alpha, \beta, \gamma \ge 0$ define the norms

$$\|g\|_{X^{\alpha,\beta}} = \sup_{0 \le t \le 1} \|g(t, .)\|_{B^{\alpha,\beta}_{x,v}(\mathbb{R}^6)},$$

$$\|F\|_{Y_\gamma} = \sup_{0 \le t \le 1} t^{-\gamma} \|F(t, .)\|_{L^\infty(\mathbb{R}^3)},$$

define the Banach space

$$Z := X^{\alpha,\beta} \times Y_\gamma, \qquad \text{with} \qquad \|(h, G)\|_Z = \max\big(\|h\|_{X^{\alpha,\beta}}, \|G\|_{Y_\gamma}\big),$$

and denote by Γ_R the ball of radius R.

In the sequel we fix

$$\gamma = a/3 - \varepsilon, \qquad \alpha = a, \qquad \beta = 1,$$

for some $\varepsilon \ll 1$.

The end of this subsection is devoted to the proof of Proposition 5.1. We assume that $g_0 \in B^{a,a}$, for some $a > 1/2$. In the sequel, we write $K = K_0$.

Step1: Φ maps some ball $\Gamma_R \subset Z$ into itself

- Firstly, we estimate $\Phi_1(g, F)$ in $X^{0,1}$. By (2.13), we have for all $0 \le t \le 1$

$$\begin{aligned} \|\Lambda_v \Phi_1(g, F)(t)\|_B &\le \varepsilon_0 \int_0^t \|\Lambda_v \mathrm{e}^{-(t-s)K} F(s)\partial_v\big(\mathrm{e}^{-sK} f_0 + g(s)\big)\|_B ds \\ &\le C\varepsilon_0 \int_0^t \|F\|_{L^\infty(\mathbb{R}^3)} \|\Lambda_v \mathrm{e}^{-(t-s)K}\|_B \|\Lambda_v\big(\mathrm{e}^{-sK} f_0 + g(s)\big)\|_B ds, \end{aligned} \tag{5.2}$$

and we estimate each factor in the previous integral thanks to the low regularity subsection results.

<u>Estimation of $\|\Lambda_v\big(\mathrm{e}^{-sK} f_0 + g(s)\big)\|_B$</u>: To begin with, we use point (i) of Corollary 3.16 to estimate $\|\Lambda_v \mathrm{e}^{-sK} f_0\|_B$. Since $f_0 \in B^{a,a}$ for some $a > 1/2$, then for $\delta = 1/2 - a/2$ we have

$$\|\Lambda_v \mathrm{e}^{-sK} f_0\|_B \le C s^{-\delta} \|f_0\|_{B^a}.$$

This gives for $0 \leq s \leq t \leq 1$

$$\begin{aligned} \|\Lambda_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)\|_B &\leq \|\mathrm{e}^{-sK}f_0\|_{B^{0,1}}+\|g(s)\|_{B^{0,1}} \\ &\leq C+Cs^{-\delta}\|f_0\|_{B^a}+\|g\|_{X^{a,1}} \\ &\leq Cs^{-\delta}(1+\|g\|_{X^{a,1}}). \end{aligned} \tag{5.3}$$

<u>Estimation of $\|F\|_{L^\infty(\mathbb{R}^3)}$</u>: By definition of the space Y_γ we have

$$\|F\|_{L^\infty(\mathbb{R}^3)} \leq s^\gamma \|F\|_{Y_\gamma}. \tag{5.4}$$

<u>Estimation of $\|\Lambda_v \mathrm{e}^{-(t-s)K}\|_{B\to B}$</u>: By point (ii) in Proposition 3.15 we have

$$\|\Lambda_v \mathrm{e}^{-(t-s)K}\|_{B\to B} \leq C(t-s)^{-1/2}. \tag{5.5}$$

Therefore by (5.2), (5.3), (5.4) and (5.5), we have

$$\begin{aligned} \|\Lambda_v \Phi_1(g,F)(t)\|_B &\leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}\int_0^t s^{\gamma-\delta}(t-s)^{-1/2}ds \\ &\leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma} t^{\gamma-\delta+1/2}. \end{aligned}$$

As a consequence (using that $\gamma-\delta+1/2 \geq 0$) we have proved

$$\|\Phi_1(g,F)\|_{X^{0,1}} \leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma} \leq C\varepsilon_0\big(1+\|(g,F)\|_Z\big)^2. \tag{5.6}$$

• We estimate $\Phi_1(g,F)$ in $X^{a,0}$. With the same arguments and the bound given in point (iii) of Corollary 3.16, for all $t \geq 0$ we obtain

$$\begin{aligned} \|\Lambda_x^a \Phi_1(g,F)(t)\|_B &\leq \varepsilon_0 \int_0^t \|\Lambda_x^a \mathrm{e}^{-(t-s)K}F(s)\partial_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)\|_B ds \\ &\leq C\varepsilon_0 \int_0^t \|F\|_{L^\infty(\mathbb{R}^3)}\|\Lambda_x^a \mathrm{e}^{-(t-s)K}\|_B \|\Lambda_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)\|_B ds \\ &\leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}\int_0^t s^{\gamma-\delta}\big((t-s)^{-3a/2}+1\big)ds \\ &\leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}\big(t^{\gamma-\delta+1-3a/2}+t^{\gamma-\delta+1}\big). \end{aligned} \tag{5.7}$$

This in turn implies (observing that $\gamma-\delta+1-3a/2>0$ provided that $a<3/4$)

$$\|\Phi_1(g,F)\|_{X^{a,0}} \leq C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma} \leq C\varepsilon_0\big(1+\|(g,F)\|_Z\big)^2. \tag{5.8}$$

• We turn to the estimation of $\|\Phi_2(g,F)\|_{Y_\gamma}$. We apply (3.16) and (3.10) with

$$h_0 = \big(e^{-tK}-1\big)f_0+\varepsilon_0\int_0^t \mathrm{e}^{-(t-s)K}F(s)\partial_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)ds,$$

then for all $0 \leq t \leq 1$

$$\|\Phi_2(g,F)(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 t^{a/3-\varepsilon}+C\varepsilon_0\int_0^t \|\Lambda^{1/2+\varepsilon}\mathrm{e}^{-(t-s)K}F(s)\partial_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)\|_B ds,$$

where we used Lemma 3.18.

To control the second term, we can proceed as in (5.7) with a replaced by $1/2+\varepsilon$. Actually we have

$$\|\Phi_2(g,F)(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 t^{a/3-\varepsilon}+C\varepsilon_0\int_0^t \|F\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{1/2+\varepsilon}\mathrm{e}^{-(t-s)K}\|_B\|\Lambda_v\big(\mathrm{e}^{-sK}f_0+g(s)\big)\|_B ds,$$

and we get

$$\begin{aligned}\|\Phi_2(g,F)(t)\|_{L^\infty(\mathbb{R}^3)} &\leq C_0 t^{a/3-\varepsilon} + C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}\int_0^t s^{\gamma-\delta}\big((t-s)^{-\frac{3}{2}(\frac{1}{2}+\varepsilon)}+1\big)ds \\ &\leq C_0 t^{a/3-\varepsilon} + C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}\big(t^{\gamma-\delta+1/4-3\varepsilon/2}+t^{\gamma-\delta+1}\big) \\ &\leq t^\gamma\big[C_0 + C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma}(t^{-\delta+1/4-3\varepsilon/2}+t^{-\delta+1})\big],\end{aligned}$$

since $\gamma = a/3-\varepsilon$. Therefore

$$\|\Phi_2(g,F)\|_{Y_\gamma} \leq C_0 + C\varepsilon_0(1+\|g\|_{X^{a,1}})\|F\|_{Y_\gamma} \leq C\varepsilon_0\big(1+\|(g,F)\|_Z\big)^2, \tag{5.9}$$

provided that $\delta<1$ and $1/2-a/2=\delta<1/4-3\varepsilon/2$. This latter condition can be satisfied for $\varepsilon>0$ small enough, since $a>1/2$.

As a result, by (5.6), (5.8) and (5.9) there exists $C>0$ such that

$$\|\Phi(g,F)\|_Z \leq C_0 + C\varepsilon_0\big(1+\|(g,F)\|_Z\big)^2.$$

Therefore we can choose $\varepsilon_0>0$ small enough so that Φ maps the ball $\Gamma_{2C_0}\subset Z$ into itself.

Step2: Φ is a contraction of Γ_{2C_0}

With exactly the same arguments, we can also prove the contraction estimate

$$\|\Phi_1(g_2,F_2)-\Phi_1(g_1,F_1)\|_Z \leq C\varepsilon_0\|(g_2-g_1,F_2-F_1)\|_Z\big(1+\|(g_1,F_1)\|_Z+\|(g_2,F_2)\|_Z\big).$$

We do not write the details.

As a conclusion, if $\varepsilon_0>0$ is small enough, Φ has a unique fixed point in $\Gamma_{2C_0}\subset Z$. This shows the existence of a unique $g\in\mathcal{C}\big([0,1]\,;\,B^{a,1}(\mathbb{R}^6)\big)$ such that $f=\mathrm{e}^{-tK}f_0+g$ solves (1.1).

5.2. Long time analysis: $t\in]0,+\infty[$. — We now study long time existence and trend to equilibrium. We use here the spaces defined in the Subsection 4.1. Let $1/2<\beta<1$ and $0<\alpha<2/3$ be such that $\alpha<(1+\beta)/3$. Fix also

$$0<\delta<\beta/2-1/4, \qquad 0<\sigma\leq\frac{1}{2}\min\big(1-\beta+\alpha,1\big),$$

which is realised for, say, $\sigma=1/12$. From now, we assume that all these conditions are satisfied.

In this section we prove the following result

Proposition 5.2. — *Let $d=3$. Assume that $1/2<a<2/3$ and that $f_0\in B^{a,a}(\mathbb{R}^6)$. Assume moreover that Assumptions 1 and 2 are satisfied. Then if $|\varepsilon_0|$ is small enough, there exists a unique local mild solution f to (1.1) which reads*

$$f(t)=\mathcal{M}_\infty+e^{-tK}(f_0-\mathcal{M}_\infty)+h(t)$$

where $h\in X^{\alpha,\beta}_\delta$, thus

$$h\in\mathcal{C}\big(]0,+\infty[\,;\,B^{\alpha,\beta}(\mathbb{R}^6)\big).$$

Since $f_0-\mathcal{M}_\infty\in B^\perp$, and by definition of the space $X^{\alpha,\beta}_\delta$, we obtain the exponentially fast convergence of f to $\mathcal{M}_\infty$. Notice that in the previous result, the parameters (α,β) can be chosen independently from a. If one chooses $\alpha=a$ and β close to 1, then the result of Proposition 5.2 combined with Proposition 5.1 and Corollary 3.2 implies Theorem 1.3.

We now turn to the proof of Proposition 5.2. Let $g_0:=f_0-\mathcal{M}_\infty\in B^{a,a}\cap B^\perp$, for some $a>1/2$. We denote by Γ_1 the unit ball in Z, and in the sequel, we use the same notations and decomposition as in Section 4.1. Then

Step1: Φ maps the ball $\Gamma_1 \subset Z$ into itself

• Firstly, we estimate $\Phi_1(h,G)$ in $X^{0,\beta}_\delta$. By (2.13), we have for all $t \geq 0$

$$\begin{aligned}\|\Lambda_v^\beta \Phi_1(h,G)(t)\|_B &\leq \varepsilon_0 \int_0^t \|\Lambda_v^\beta \mathrm{e}^{-(t-s)K}\big(F_0+G\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK}g_0 + h(s)\big)\|_B\,ds \\ &\leq C\varepsilon_0 \int_0^t \|F_0+G\|_{L^\infty(\mathbb{R}^3)}\|\Lambda_v^\beta \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp}\|\Lambda_v^\beta\big(\mathcal{M}_\infty+\mathrm{e}^{-sK}g_0+h(s)\big)\|_B\,ds,\end{aligned} \tag{5.10}$$

and we estimate each factor in the previous integral.

<u>Estimation of $\|\Lambda_v^\beta\big(\mathcal{M}_\infty+\mathrm{e}^{-sK}g_0+h(s)\big)\|_B$</u>: To begin with, we use Lemma 2.8 to estimate the term $\|\Lambda_v^\beta \mathrm{e}^{-sK}g_0\|_B$. Since $g_0 \in B^{a,a}$ for some $a > 1/2$, there exists $0 < \delta < \beta/2 - 1/4$ such that

$$\|\Lambda_v^\beta \mathrm{e}^{-sK}g_0\|_B \leq C(1+s^{-\delta})\|g_0\|_{B^a}.$$

Then, we use that $\mathcal{M}_\infty \in B^{\alpha,\beta}$, and by (2.8) we obtain

$$\begin{aligned}\|\Lambda_v^\beta\big(\mathcal{M}_\infty+\mathrm{e}^{-sK}g_0+h(s)\big)\|_B &\leq \|\mathcal{M}_\infty\|_{B^{0,\beta}} + \|\mathrm{e}^{-sK}g_0\|_{B^{0,\beta}} + \|h(s)\|_{B^{0,\beta}} \\ &\leq C + C(1+s^{-\delta})\|g_0\|_{B^a} + (1+s^{-\delta})\|h\|_{X^{\alpha,\beta}_\delta} \\ &\leq C(1+s^{-\delta})(1+\|h\|_{X^{\alpha,\beta}_\delta}).\end{aligned} \tag{5.11}$$

<u>Estimation of $\|F_0+G\|_{L^\infty(\mathbb{R}^3)}$</u>: By (3.13) and the definition of the space Y we get for $\varepsilon > 0$ small enough and $a > 1/2$

$$\begin{aligned}\|F_0+G\|_{L^\infty(\mathbb{R}^3)} &\leq \|F_0\|_{L^\infty(\mathbb{R}^3)} + \|G\|_{L^\infty(\mathbb{R}^3)} \\ &\leq Ce^{-\kappa s/2}\|g_0\|_{B^a} + C\mathrm{e}^{-\sigma\kappa s}\|G\|_Y \\ &\leq C\mathrm{e}^{-\sigma\kappa s}(1+\|G\|_Y).\end{aligned} \tag{5.12}$$

<u>Estimation of $\|\Lambda_v^\beta \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp\to B^\perp}$</u>: This is exactly (2.10), namely

$$\|\Lambda_v^\beta \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp\to B^\perp} \leq C\big((t-s)^{-1/2}+1\big)e^{-\kappa(t-s)}.$$

Therefore by (5.10), (5.11), (5.12), we have when $\delta < 1$

$$\begin{aligned}&\|\Lambda_v^\beta \Phi_1(h,G)(t)\|_B \leq \\ &\qquad \leq C\varepsilon_0(1+\|h\|_{X^{\alpha,\beta}_\delta})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\int_0^t \big(s^{-\delta}+1\big)\big((t-s)^{-1/2}+1\big)e^{-\kappa(t-s)/2}ds\end{aligned}$$

Now, by (3.15), if we denote by $\eta = -\min(1/2-\delta, 0)$, we have

$$\begin{aligned}\|\Lambda_v^\beta \Phi_1(h,G)(t)\|_B &\leq C\varepsilon_0(1+\|h\|_{X^{\alpha,\beta}_\delta})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\big(\mathbf{1}_{\{0<t\leq 1\}}t^{-\eta}+1\big) \\ &\leq C\varepsilon_0(1+\|h\|_{X^{\alpha,\beta}_\delta})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\big(t^{-\delta}+1\big).\end{aligned} \tag{5.13}$$

As a consequence we have proved

$$\|\Phi_1(h,G)\|_{X^{0,\beta}_\delta} \leq C\varepsilon_0(1+\|h\|_{X^{\alpha,\beta}_\delta})(1+\|G\|_Y) \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2. \tag{5.14}$$

• We estimate $\Phi_1(h,G)$ in $X_\delta^{\alpha,0}$. With the same arguments and the bound (2.9), for all $t \geq 0$ we obtain

$$
\begin{aligned}
(5.15)\quad & \|\Lambda_x^\alpha \Phi_1(h,G)(t)\|_B \leq \\
&\leq \varepsilon_0 \int_0^t \|\Lambda_x^\alpha \mathrm{e}^{-(t-s)K}\big(F_0+G\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big)\|_B ds \\
&\leq C\varepsilon_0 \int_0^t \|F_0+G\|_{L^\infty(\mathbb{R}^3)} \|\Lambda_x^\alpha \mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp} \|\Lambda_v^\beta\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big)\|_B ds \\
&\leq C\varepsilon_0 (1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t} \int_0^t \big(s^{-\delta}+1\big)\big((t-s)^{-1/2+\beta/2-3\alpha/2}+1\big)e^{-\kappa(t-s)/2} ds \\
&\leq C\varepsilon_0 (1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\big(t^{-\delta}+1\big),
\end{aligned}
$$

using (3.15) and the fact that $1/2+\beta/2-3\alpha/2>0$.

This in turn implies

$$
(5.16)\qquad \|\Phi_1(h,G)\|_{X_\delta^{\alpha,0}} \leq C\varepsilon_0(1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y) \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2.
$$

• We turn to the estimation of $\|\Phi_2(h,G)\|_Y$. We apply (3.10) with

$$
h_0 = \varepsilon_0 \int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s)+G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big) ds,
$$

then for all $t \geq 0$

$$
\begin{aligned}
\|\Phi_2(h,G)(t)\|_{L^\infty(\mathbb{R}^3)} &\leq C\varepsilon_0 \|\Lambda^{1/2+\varepsilon} \int_0^t \mathrm{e}^{-(t-s)K}\big(F_0(s)+G(s)\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big) ds\|_B \\
&\leq C\varepsilon_0 \int_0^t \|\Lambda^{1/2+\varepsilon}\mathrm{e}^{-(t-s)K}\big(F_0+G\big)\partial_v\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big)\|_B ds.
\end{aligned}
$$

Now we can proceed as in (5.15) with α replaced by $1/2+\varepsilon$. Actually we have

$$
\begin{aligned}
&\|\Phi_2(h,G)(t)\|_{L^\infty(\mathbb{R}^3)} \leq \\
&\qquad \leq C\varepsilon_0 \int_0^t \|F_0+G\|_{L^\infty(\mathbb{R}^3)} \|\Lambda^{1/2+\varepsilon}\mathrm{e}^{-(t-s)K}\Lambda_v^{1-\beta}\|_{B^\perp} \|\Lambda_v^\beta\big(\mathcal{M}_\infty + \mathrm{e}^{-sK} g_0 + h(s)\big)\|_B ds,
\end{aligned}
$$

and for $1/2<\beta<1$, $\varepsilon \ll 1$, by (2.9) and (3.15) we get

$$
\begin{aligned}
&\|\Phi_2(h,G)(t)\|_{L^\infty(\mathbb{R}^3)} \leq \\
&\quad \leq C\varepsilon_0 (1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t} \int_0^t \big(s^{-\delta}+1\big)\big((t-s)^{-\frac{1}{2}+\frac{\beta}{2}-\frac{3}{2}(\frac{1}{2}+\varepsilon)}+1\big)e^{-\kappa(t-s)/2} ds \\
&\quad \leq C\varepsilon_0 (1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t}\big(\mathbf{1}_{\{0<t\leq 1\}} t^{-1/4+\beta/2-\delta-3\varepsilon/2}+1\big) \\
&\quad \leq C\varepsilon_0 (1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y)\mathrm{e}^{-\sigma\kappa t},
\end{aligned}
$$

if $\varepsilon>0$ is chosen small enough so that we have $\delta<\beta/2-1/4-3\varepsilon/2$. This in turn implies

$$
(5.17)\qquad \|\Phi_2(h,G)\|_Y \leq C\varepsilon_0(1+\|h\|_{X_\delta^{\alpha,\beta}})(1+\|G\|_Y) \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2.
$$

As a result, by (5.14), (5.16) and (5.17) there exists $C>0$ such that

$$
\|\Phi(h,G)\|_Z \leq C\varepsilon_0\big(1+\|(h,G)\|_Z\big)^2.
$$

Therefore we can choose $\varepsilon_0>0$ small enough so that Φ maps the ball $\Gamma_1 \subset Z$ into itself.

Step2: Φ is a contraction of Γ_1

With exactly the same arguments, we can also prove the contraction estimate

$$\|\Phi_1(h_2, G_2) - \Phi_1(h_1, G_2)\|_Z \leq C\varepsilon_0 \|(h_2 - h_1, G_2 - G_1)\|_Z \big(1 + \|(h_1, G_1)\|_Z + \|(h_2, G_2)\|_Z\big).$$

We do not write the details.

As a conclusion, if $\varepsilon_0 > 0$ is small enough, Φ has a unique fixed point in $\Gamma_1 \subset Z$. This shows the existence of a unique $h \in \mathcal{C}\big(\,]0, +\infty[\,;\, B^{\alpha,\beta}(\mathbb{R}^6)\big)$ such that $f = \mathcal{M}_\infty + \mathrm{e}^{-tK} g_0 + h$ solves (1.1).

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